The geometrical basis of arithmetical knowledge: Frege & Dehaene*

Sorin Costreie

Received: 18/09/2017
Final version: 02/04/2018

BIBLID 0495-4548(2018)33:2p.361-370
DOI: 10.1387/theoria.18089

ABSTRACT: Frege writes in *Numbers and Arithmetic* about *kindergarten-numbers* and "an *a priori* mode of cognition" that they may have "a geometrical source." This resembles recent findings on arithmetical cognition. In my paper, I explore this resemblance between Gottlob Frege's later position concerning the geometrical source of arithmetical knowledge, and some current positions in the literature dedicated to arithmetical cognition, especially that of Stanislas Dehaene. In my analysis, I shall try to mainly see to what extent (Frege's) *logicism* is compatible with (Dehaene's) *intuitionism*.

Keywords: logicism, intuitionism, Frege, Dehaene, arithmetical cognition.

RESUMEN: En *Numbers and Arithmetic* Frege escribe que *kindergarten-numbers* y «un modo *a priori* de cognición» pueden tener «un origen geométrico». Esto se asemeja a algunos descubrimientos recientes sobre cognición aritmética. En mi artículo, exploro la semejanza entre la útlima posición de Gottlob Frege acerca del origen geométrico del conocimiento aritmético, y algunas posiciones actuales en la bibliografía sobre cognición aritmética, especialmente la de Stanislas Dehaene. En mi análisis, intento principalmente determinar hasta qué punto el *logicismo* (de Frege) es compatible con el *intuicionismo* (de Dehaene).

Palabras clave: logicismo, intuicionismo, Frege, Dehaene, cognición aritmética.

We commonly think of arithmetic as an abstract and non-spatial domain, populated by numbers which are (a kind of) logical objects that are far from possessing any geometrical traits. Moreover, besides those abilities required for the cognitive manipulation of symbols, no spatial processing seems to be implicated in arithmetical cognition. However, at a closer analysis of the phenomena involved in mathematical cognition, we may discover that spatial and number processing are intimately connected, and thus that geometrical and arithmetical knowledge share a common epistemic foundation. Hence this paper is part of a larger project which tries to reconcile traditional philosophies of mathematics, commonly seen as mutually incompatible, and integrate them with contemporary research.

For most of his life, Gottlob Frege held that geometry and arithmetic should be kept separate, because they have different epistemic foundations. However, by the end of his life,

* Versions of this material were presented at two workshops: *The Cognitive Basis of Logico-Mathematical Knowledge* (Bergen, November 2015) and *From Basic Cognition to Mathematical Practice* (Seville, September 2016). I thank those audiences for their comments and critical feedback, and I also thank two anonymous referees for this journal, for their helpful comments and suggestions on an earlier draft. I also thank Sorin Chiper and Alina Tigau for helping me revising this paper.

 BY-NC-ND

he came to the conclusion that the two are intimately connected, and that arithmetic seems to have "a geometrical source", which is both objective and *a priori*. This claim is substantially supported by recent findings about arithmetical cognition. The main aim of my article is to present and analyze this resemblance between Gottlob Frege's later position concerning the geometrical source of arithmetical knowledge, and some current views in the literature dedicated to arithmetical cognition, especially that of Stanislas Dehaene concerning our *number sense*. My analysis is twofold: first, I will investigate whether Frege's *logicism* is compatible with Dehaene's *intuitionism* and second, I will attempt to see in which sense we may argue that arithmetical cognition has a geometrical epistemic basis. Following these aims, the paper is structured in four sections: Section 1 gives an overview of Frege's initial logicism; Section 2 presents Dehaene's intutionism; Section 3 tackles Frege's later logicism or his *intu-logicism* and finally, Section 4 concludes by briefly discussing the role of geometrical knowledge in arithmetical cognition.

1. *Frege's initial logicism*

As is commonly acknowledged, Gottlob Frege's lifetime project was *logicism*, yet this project was not just *logicism per se*; for *logicism* as the formal thesis that arithmetic is (reducible to) logic is compatible with both *formalism* and *psychologism*, whereas Frege constantly and explicitly rejects both. For Frege, arithmetical signs have content (contrary to *formalism*) and this content is not subjective (contrary to *psychologism*). To put it briefly, for Frege, the subject matter of arithmetic is *contentful*, i.e. it is not a mere game with empty signs, and its content is *objective*. This objective content of mathematics is in Frege's view a crucial characteristic of arithmetic, one that explains its universal applicability and lends it the status of a general science; thus for him the most dangerous threat to a proper characterization of arithmetic is to consider that it lacks any form of proper content, i.e. to endorse mathematical formalism. Frege's logicist project, started mainly as an epistemic reaction to both Kant's intuitionistic position and to the growing formalist trend among his contemporary mathematicians, especially that exemplified and defended by his colleagues Eduard Heinrich Heine (University of Halle) and Carl Johannes Thomae (University of Jena). However, Frege's logicist project was labeled and received mainly as Platonism, i.e. as an ontological position. What is interesting is that at the end of his life, Frege loosened his ontological Platonist commitments precisely retaining his initial epistemic stance that arithmetic is logic. The change boiled down to the addition of the idea that arithmetic has a geometric conceptual foundation, thus making room for "an *a priori* mode of cognition", fundamental to all mathematical knowledge.

In his *On Formal Theories of Arithmetic*,[1] Frege identifies two ways in which arithmetic could be seen as 'formal'. The former way is given by the "logical or formal nature of arithmetic", namely the fact that "all arithmetical propositions can be derived from definitions alone using purely logical means (...) in direct contrast with geometry." (Frege 1984, 112)

This could be regarded today as a *syntactic* characterization of arithmetic as being 'formal', and, for Frege, this is the *good* formalism in mathematics. The main purpose of Frege's

[1] In Frege (1984).

begriffsschrift was to facilitate this formal derivation of arithmetical truths from the logical ones.[2] It is worth mentioning here that Frege sees arithmetic "in direct contrast with geometry." For him, only arithmetic was regarded as analytic (as reducible to logic), whereas geometry, given its appeal to the intuition of space, is not, for intuitions are subjective.

The latter way is characterized by Frege as problematic and should consequently be vigorously refuted. This is the *semantic* characterization of arithmetical formalism which regards arithmetical symbols as empty or meaningless —"in which case we should have neither truths, nor a science, of arithmetic."[3] For Frege, this is the *bad* formalism, and it is characterized by the lack of the crucial distinction between signs and their contents, which should be definitely defeated in arithmetic —a science with its own content.

The 'good formalism' is exemplified in fact by Frege's *logicism*, whereas the 'bad formalism' is one of the main enemies of logicism, i.e. *mathematical formalism,* especially those early versions of Heine and Thomae. In a footnote to *Grundlagen* Frege argues explicitly that his *begriffsschrift* is "designed to be capable of expressing not only the logical form, like Boole's notation, but also the *content* of a proposition."[4] This *content*, which will be further split into sense (*Sinn*) and reference (*Bedeutung*),[5] is crucial to Frege; it is a sort of objective logical information that is 'carried' in the course of inferences of mathematical proofs, and that it should be kept pure in relations with intuitions, which are subjective and personal.

Thus, a *begriffsschrift* has a dual role: first, to prevent any infiltration of *subjective* intuitive elements into the proofs of the system, and second, to carry on *objective* semantic information. The first role is secured by the logical deductive formalism of the system, whereas the second role is provided by his conception of the *begriffsschrift* as a *characteristica universalis* and not just as Boole's *calculus racinator.* It is worth noting that *logicism* (understood strictly as the thesis that arithmetic is reducible to logic) can be made compatible with (the views of the nineteenth century) *formalism*. It is also worth noting that *logicism* (understood strictly as the thesis that mathematics —or at least arithmetic— is reducible to logic) can be made compatible with *intuitionism*, if by 'logic' we mean an (sort of) intuitionistic acceptable logic.

The insoluble challenge is to try to make all three positions compatible at once. This does not work because of the inner tension between *formalism* and *intuitionism*. We could not consistently hold that mathematics is completely formal, i.e. totally contentless, and simultaneously maintain its intuitionistic character, namely that its content is based on pure intuitions. There is an insurmountable inner tension between the two which makes them mutually incompatible. So far, the moral is that *logicism*, as a pure thesis concerning the nature of mathematics, is compatible with each *formalism* and *intuitionism*, yet not with both at once. However, Frege's logicist project is basically at odds with *formalism*, for Frege's logicism is not just characterized by the thesis that arithmetic is logic, but it comprises another important aspect, namely the thesis that arithmetic is contentful. This fundamental claim is responsible for the persistent Fregean rejection of *formalism.* But could Frege's *logicism* be seen as being compatible with (a form of) *intuitionism*? Surprisingly, the straight

---

[2] By *begriffsschrift* I mean Frege's logical system which has presented for the first time in his work *Begriffsschrift*. See Frege (1879).

[3] Frege (1984, 114).

[4] Gl §91; Frege (1884, 103).

[5] See Frege (1892).

answer is yes; as far as this form of *intuitionism* does not rely of subjective intuitions, I think that Frege's *logicism*[6] does not automatically reject it, and it could consequently be seen as compatible with a form of intuitionism.

2. Dehaene's intutionism

Let us now tackle a form of modern mathematical intuitionism. Stanislas Dehaene's *intuitionism* is based on his *number sense* assumption[7] —the peculiar idea that we owe our mathematical intuitions to an inherited capacity which we share with other animals, namely, the rapid perception of approximate numbers of objects. But, first of all, why should we characterize Dehaene's position as *intuitionistic*? Basically, because he explicitly acknowledges this: "among the available theories on the nature of mathematics, *intuitionism* seems to me to provide the best account of the relations between arithmetic and the human brain." (Dehaene 2011, 226)

What is particularly interesting to us here, in the context of Frege's late philosophical ideas, is the fact that Dehaene does not characterize this intuition as *subjective*: "I believe that most mathematicians do not just manipulate symbols according to purely *arbitrary* rules. On the contrary, they try to capture in their theorems certain physical, numerical, geometrical, and logical intuitions." (Dehaene 2011, 226)

Moreover, an important feature of mathematics is exactly this logical formalization of our common intuitions:

> "Mathematics consists in the formalization and progressive refinement of our fundamental intuitions. As humans, we are born with multiple intuitions concerning numbers, sets, continuous quantities, iteration, logic, and the geometry of space. Mathematicians strive to formalize these intuitions and turn them into logically coherent systems of axioms." (Dehaene 2011, 228)

These intuitions are not subjective, for they are not personal. It is like we share a common stock of *a priori* intuitions which guide us and shape our arithmetical thinking; "to affirm that arithmetic is the product of the human mind does not imply that it is *arbitrary* and that, on some other planet, we might have been born with the idea that 1 + 1 = 3." (Dehaene 2011, 231)

The crucial characteristic of these *a priori* intuitions is exactly the fact that they are not *arbitrary*. For Dehaene, we are born with a kind of arithmetical pre-settings, which shape our early mathematical knowledge: "Throughout phylogenetic evolution, as well as during cerebral development in childhood, selection has acted to ensure that the brain constructs *internal representations* that are adapted to the external world. Arithmetic is such an adaptation." (Dehaene 2011, 231)

But what are these *internal representations*, and in which way could we eventually say that arithmetical cognition is grounded on geometrical representations? Dehaene is not

---

[6] For more about this possible compatibility between logicism and intuitionism, and the way in which Frege's fight against formalism strongly motivated *Frege's puzzle* and the introduction of his sense/reference distinction, see Costreie (2013).

[7] Extensively presented in Dehaene (2011).

very precise about what they are, yet the best supporting idea in this sense may be offered by his so-called *Spatial-Numerical Association of Response Codes*, or the SNARC effect, which shows us a robust link between the concept of number and space.

The SNARC Effect was presented and discussed for the first time in Dehaene *et al.* (1993), and it suggests that people represent numbers in the form of an *imaginary number line*. The experiment consists in asking (adult) subjects to classify numbers as smaller or larger than 65. To this end, they hold two response keys, one in the left hand and the other in the right hand. The result is that the subjects respond faster to large numbers with their right hand, and faster to small numbers with their left hand. Dehaene *et al.* claim that this must be because respondents are imaginarily place numbers on a number line, where smaller numbers are always to the left.

Based on these findings and on similar others, in his famous book *The Number Sense*, Dehaene developed an intuitionistic position regarding mathematics, while rejecting mathematical formalism:

> "If mathematics is nothing more than a formal game, how is it that it focuses on specific and universal categories of the human mind such as numbers, sets, and continuous quantities? Why do mathematicians judge the laws of arithmetic to be more fundamental than the rules of chess?" (Dehaene 2011, 225)

Interestingly enough, Dehaene even resorts to the same example used by Frege himself in order to discuss and reject the formalist position,[8] namely that of the parallelism between mathematics and chess. For Frege and Dehaene, adopting the formalist stance in mathematics, would transform and regard the whole of mathematics as a mere game of chess, which is for both preposterous; mathematics applies to the real world, whereas chess does not. Mathematical objects are neither invented following pure conventions, nor are they mere symbols manipulated according to arbitrary rules.

Dehaene explicitly argues along with Kant and Poincaré that

> "[m]athematical objects are fundamental, a priori categories of human thought that the mathematician refines and formalizes. The structure of our mind forces us, in particular, to parse the world into discrete objects; this is the origin of our intuitive notions of set and of number. [...] Among the available theories on the nature of mathematics, intuitionism seems to me to provide the best account of the relations between arithmetic and the human brain. The discoveries of the latest few years in the psychology of arithmetic have supplied new arguments to support the intuitionist view that neither Kant nor Poincaré could have known. These empirical results tend to confirm Poincare's postulate that number belongs to the "natural objects of thought," the innate categories according to which we apprehend the world." (Dehaene 2011, 226-7)

However, Dehaene does not endorse Brouwer's version of intuitionism, for it makes the whole construction of mathematics too subjective and personal, which is also exactly the reason why Frege rejected psychologism in mathematics. This form of subjective mental constructivism does not explain the objective applicability of mathematics onto the physical world, which is an essential Fregean characteristic for a contentful science as mathematics.

---

8 In Gg §§88-137, Frege is discussing *in extenso* exactly this example of Thomae that mathematics is no more than a game of chess. See Frege (1960).

3. *Frege's late logicism or his* intu-logicism[9]

Let us now go back to Frege on the relation between continuous and discrete quantities. Regarding that point, Frege argues in *Numbers and Arithmetic* that "numbers of different kinds have arisen in different ways and must be distinguished accordingly" (Frege, 1979, 276). For him, we have those *kindergarten-numbers*, which are basically discrete natural numbers, learned empirically —"drilled into children by parents and teachers." He continues his characterization of kindergarten-numbers that

> "[i]n this way something like images of numbers are formed in the child's mind. But this is an artificial process which is imposed on the child rather than one which develops naturally within him. But even if it were a natural process, there would be hardly anything to learn about the real nature of the kindergarten numbers from the way they originate psychologically." (Frege 1979, 276)

Here, the suggestion is clear: we may talk about a way in which we learn artificially and empirically about discrete natural numbers, yet the epistemic origin and its cognitive foundation is different, and it "develops naturally within him" on the basis of the apprehension of continuous magnitudes. This is so because, for Frege, the *kindergarten numbers* are "extremely limited in their application", as they could not explain the conceptual construction and epistemic grasp of important categories in the realm of numbers, i.e. those of irrational numbers: "The labours of mathematicians have indeed led to another kinds of numbers, to the *irrationals*." As he continues, the problem is that "there is no bridge which leads across from *kindergarten-numbers* to the *irrationals*." (Frege 1979, 276)

Frege thus acknowledges his initial mistake, when he thought that he could build the whole of arithmetic on *kindergarten numbers*, i.e. natural numbers learned through education, making associations between numbers and various suggestive representations of them:

> I myself at one time held it to be possible to conquer the entire number domain, continuing along a purely logical path from the *kindergarten numbers*; I have seen the mistake in this. I was right in thinking that you cannot do this if you take an *empirical route*. (Frege 1979, 276)

The 'empirical route' was eventually refuted by Frege in *Logic in Mathematics*, for it could not deal with infinite numbers. There is no inner representation for them, so he concludes that the arithmetic "cannot be based on sense perception." Thus it cannot be *a posteriori*: "So an *a priori* mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles as I originally assumed. There is the further possibility that it has a *geometrical source*." (Frege 1979, 227)

That may be seen as a substantial departure from his initial logicist stance, where arithmetic and geometry were thought to have different epistemic foundations: "The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis —a geometrical one in fact— so that mathematics in its entirety is really geometry." (Frege 1979, 227)

[9] By 'intu-logicism' I understand an oxymoron formed by *intuitionism* and *logicism*, two *prima facie* incompatible positions, yet, as I have showed, this is a label for Frege's later version of his logicist position, where he made room for objective intuitions, as an *a priori* geometrical epistemic basis for arithmetical cognition.

However, the flaw of the original mistake was that he somehow associated all kinds of intuitions with *subjective* representations, leaving no room for any kind of *objective* representations of mathematical numbers. At the end of his life, he realized the he had thrown out the baby with the bathwater. The "baby" is this *a priori* mode of cognition that has a *geometrical source*. Being *a priori* does not necessarily imply being subjective, and this was Frege's initial mistake, which he corrected at the end of his life. This objective *a priori* mode of mathematical knowledge is nicely presented by Dehaene as well, and this is why I think his position is paradigmatic for a kind of intuitionism constructed on *objective* intuitions. This is crucial for Frege, for thus he may still defend the logicist project and apprehend the infinite through an *objective a priori* mode of cognition:

> "From the geometrical source of knowledge flows the infinite in the genuine and strictest sense of this word. [...] We have infinitely many points on every interval of a straight line. [...] We cannot imagine the totality of these. [...] One man may be able to imagine more, another less. But here we are not in the domain of psychology, of the imagination, of what is subjective, but in the domain of the objective, of what is true." (Frege 1979, 273)
>
> This is why for Frege geometry and philosophy, not just arithmetic and logic, are intimately intertwined: "it is here that *geometry* and *philosophy* come closest together. In fact they belong to one another. A philosopher who has nothing to do with geometry is only half a philosopher, and a mathematician with no element of philosophy in him is only half a mathematician. These disciplines have estranged themselves from one another to the detriment of both." (Frege 1979, 273)

It is this geometrical source of knowledge, both *a priori* and objective, which offers us the foundation for any further mathematical knowledge, we may thus conceptually grasp infinity, which is otherwise left outside sensorial perception: "it is evident that sense perception can yield nothing infinite. However many stars we may include in our inventories, there will never be infinitely many. [...] For this we need a special source of knowledge, and one such is the *geometrical*." (Frege 1979, 274).

4. *The role of geometrical knowledge in arithmetical cognition*

Turning back to Dehaene, there are passages in his *The Number Sense* that sound strikingly similar to Frege's later position:

> "The foundations of any mathematical construction are grounded on fundamental intuitions such as notions of set, number, space, time, or logic. These are almost never questioned, so deeply do they belong to the irreducible representations concocted by our brain. Mathematics can be characterized as the progressive formalization of these intuitions. Its purpose is to make them more coherent, mutually compatible, and better adapted to our experience of the external world." (Dehaene 2011, 228)

Moreover, yet rather unsurprisingly, Dehaene, exactly like Frege, places himself further somehow in between *Platonism* and *intuitionism*:

> "The hypothesis of a partial adaptation of mathematical theories to the regularities of the physical world can perhaps provide some grounds for a reconciliation between Platonists and intuitionists. Platonism hits upon an undeniable element of truth when it stresses that physical reality is organized according to structures that predate the human mind. [...] Numbers, like other

> mathematical objects, are mental constructions whose roots are to be found in the adaptation of the human brain to the regularities of the universe." (Dehaene 2011, 233)

The apparent conclusion of both Frege and Dehaene may be thus clearly stated —as this 'concreteness' of mathematical cognition in relation with the real world: "When we think about numbers, or do arithmetic, we do not rely solely on a purified, ethereal, abstract concept of number. Our brain immediately links the abstract number to concrete notions of size, location and time. We do not do arithmetic "in the abstract."' (Dehaene 2011, 246)

The moral so far is that it seems that both Frege and Dehaene claim that there should be a kind of geometrical epistemic basis of the knowledge of numbers. This idea is endorsed by Dehaene's intuitionism, which holds that arithmetical cognition is based on and evolves from a kind of objective geometrical *a priori* knowledge. Let us recall the SNARC effect. This result is of key importance for the current issue of the spatial coding of numbers. It demonstrates that numerical magnitude information is spatially encoded. Further experiments conducted in this direction endorse this hypothesis and indicate that:

(i) The experiment works for negative numbers as well.[10]
(ii) The direction of the line is culturally dependent.[11]
(iii) Its origin is of a conceptual rather than visuospatial nature.[12]

What is important here is to see that we commonly order numbers along an imaginary line, which suggests that in order to mentally process numbers, we need to previously arrange them in a line. This suggests the existence of an interesting fact concerning mathematical cognition, namely that we first epistemically assimilate continuous lines, and only afterwards identify discrete points (on them).

What other contemporary findings support these ideas? Further evidence may come from studies concerning *Gerstmann's syndrome*.[13] These impairments might obstruct the use of writing multi-digit numerals and forming visual images of them (as well as of a number line), thereby preventing mental calculation using the place system of numerals. It looks like due to a physical impairment of the brain, which basically affects our capacity to 'visualize lines', we could not rightly process any more numbers.[14] This strongly suggests that the two types of mathematical knowledge, which have traditionally been regarded as different —arithmetical and geometrical— may be underpinned by the same epistemic *a priori* basis which, in fact, endorses all mathematical knowledge, and which may be characterized as being 'geometrical' in nature. All these ideas are now endorsed by recent findings in the field of neurosciences:

> "[n]euroimaging has confirmed and greatly elaborated the findings from neurological patients. It suggests that the IPS (*intraparietal sulcus*) is the locus of core numerical processing. [...] The IPS and surrounding regions also respond to tasks in which magnitudes such as time, size and velocity are analyzed, and it has been suggested that numerical information emerges from a generalized magnitude system." (Butterworth & Walsh 2011, 619)

---

[10] Shaki & Petrusic (2005).

[11] Zebian (2005).

[12] Fias & van Dijk & Gevers (2011).

[13] In 1940, a German neurologist, Josef Gerstmann, showed that a lesion of the left inferior parietal region can cause a tetrad of deficits: dyscalculia, finger agnosia, dysgraphia, and left-right confusion. For more, see Gerstmann (1940).

[14] For more, see Dehaene & Cohen (1997).

Many numerate people, approximately 15%, form a mental image of the sequence of numbers, called by Sir Francis Galton *number forms*,[15] where the sequence is represented in two dimensions. Number forms are described by Dehaene (2011) as follows:

> *Number forms* can be likened to a conscious and enriched version of the mental number line that we all share. While most people's mental number line is apparent only in subtle reaction time experiments, number forms are readily available to awareness and are also richer in visual details, such as color or a precise orientation in space." (Dehaene 2011, 73)

Moreover, for some people, numbers are not only mentally placed on an imaginary line, but they are colored and are placed in specific spatial locations:

> "[t]hough a majority of people have an unconscious mental number line oriented from left to right, some have a much more vivid image of numbers. Between 5% and 10% of humanity is thoroughly convinced that numbers have colors and occupy very precise locations in space." (Dehaene 2011, 71)

Thus it seems that there are some a *priori* pre-settings of our mind, which help us deal with the concepts of space and time, and which consists of a kind of prerequisites for all kind of mathematical knowledge.

I would like to conclude with Einstein's description of his inner thinking creative processes, which is also cited by Dehaene (2011), in order to exemplify the role of language and intuitions in mathematics:

> "Words and language, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thought are certain *signs* or *images*, more or less clear, that I can reproduce and recombine at will." (Dehaene 2011, 136)

Summing up, we have tried so far to see in what way Frege's later *logicism* may be compatible with a form of mathematical intuitionism, and one of the most suitable candidates on the market seems to be Dehaene's *intuitionism*. Both share the idea that arithmetical thinking has a geometrical epistemic source, which is equally *objective* and *a priori*. This is an interesting result which is endorsed by many contemporary studies and findings in the area of arithmetical cognition, and which eventually shows that arithmetical knowledge is developed ultimately on a geometrical epistemic foundation.

REFERENCES

Beaney, Michael. ed. 1997. *The Frege reader*. Oxford: Blackwell.

Butterworth, Brian and Vincent Wlash. 2011. Neural basis of mathematical cognition. *Current Biology* 21/16: 618-621.

Costreie, Sorin. 2013. Frege's puzzle and arithmetical formalism. Putting things in context. *History and Philosophy of Logic* 34/3: 207-224.

Dehaene, Stanislas, Serge Bossini and Pascal Giraux. 1993. The mental representation of parity and numerical magnitude, *Journal of Experimental Psychology General* 122/3: 371-396.

[15] A number form is a kind of mental map of numbers visualized by somebody who thinks of numbers. These forms come to one's mind automatically and involuntarily, and they were first discussed and labelled as such by Sir Francis Galton in Galton (1881).

Dehaene, Stanislas and Laurent Cohen. 1997. Cerebral pathways for calculation: Double dissociation between rote verbal and quantitative knowledge of arithmetic. *Cortex* 33/2: 219-250.

Dehaene, Stanislas. 2011. *The number sense*. Oxford: Oxford University Press.

Dehaene, Stanislas and Elisabeth Brannon. eds. 2011. *Space, time and number in the brain*. Amsterdam: Elsevier.

Feigenson, Lisa, Stanislas Dehaene and Elizabeth Spelke. 2004. Core systems of numbers. *Trends in Cognitive Science* 8/7: 307-314.

Fias, Wim, Jean-Philippe van Dijck and Wim Gevers. 2011. How is number associated with space? The role of working memory. In Stanislas Dehaene and Elisabeth Brannon, eds., *Space, time and number in the brain*, 133-148. Amsterdam: Elsevier.

Frege, Gottlob. 1879. *Begriffsschrift, eine der arithmetischen Nachgebildete Formelsprache des reinen Denkens*. Halle: I. Nebert; trans. in Gottlob Frege. *Conceptual Notations and Related Articles*. Ed. and trans. with a biography and intro. by Terell Ward Bynum. Oxford: Oxford University Press.

—. 1884. *Die Grundlagen der Arithmetik.* Trans. by J.L. Austin. *The foundations of arithmetic*. Evaston, IL: Northwestern University Press.

—. 1892. Über Sinn und Bedeutung. *Zeitschrift für Philosophie und Philosophische Kritik*, 100, 25-50; trans. in Gottlob Frege. *Translations from the philosophical writings of Gottlob Frege*. Eds. Peter Geach and Max Black. Second edition, Oxford: Blackwell.

—. 1960. *Translations from the philosophical writings of Gottlob Frege*. Eds. Peter Geach and Max Black. Second edition, Oxford: Blackwell.

—. 1962. *Grundgesetze der Arithmetik*, *begriffschtlich Abgeleitet*. One volume. Hildesheim: Georg Olms Verlagsbuchhandlung. Trans. in Gottlob Frege. 2013. *The basic laws of arithmetic: Exposition of the system*. Ed. by Philip A. Ebert and Marcus Rossberg with an introduction by Crispin Wright, and an Appendix by Roy T. Cook. Oxford: Oxford University Press.

—. 1972. *Conceptual notations and related articles*. Ed. and trans. with a biography and introduction by Terell Ward Bynum. Oxford: Oxford University Press.

—. 1979. *Posthumous writings*. Ed. by Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach; trans. by Peter Long and Roger White. Oxford: Blackwell.

—. 1984. *Collected papers on mathematics, logic and philosophy*. Ed. by Brian McGuiness, trans. by Max Black *et al.,* Oxford: Blackwell.

—. 2013. *The basic laws of arithmetic: Exposition of the system*. Trans. in Gottlob Frege. 2013. Ed. by Philip A. Ebert and Marcus Rossberg, with an introduction by Crispin Wright, and an Appendix by Roy T. Cook. Oxford: Oxford University Press.

Galton, Francis. 1881. The Visions of Sane Persons. *Popular Science Monthly* 19: 518-531.

Gerstmann, Josef. 1940. Syndrome of finger agnosia, disorientation for right and left, agraphia, and acalculia. *Archives of Neurology: Psychiatr*y 44: 398-408.

Longo, Giuseppe and Arnaud Viarouge. 2010. Mathematical intuition and the cognitive roots of mathematical concepts. *Topoi* 29: 15-27.

Shaki, Samuel and William M. Petrusic. 2005. On the mental representation of negative numbers: Context-dependent SNARC effects with comparative judgments. *Psychonomic Bulletin & Review* 12/5: 931-937.

Zebian, Samar. 2005. Linkage between number concepts, spatial thinking, and directionality of writing: the SNARC effect and the REVERSE SNARC effect in English and Arabic monoliterates, biliterates, and illiterate Arabic speakers, *Journal of Cognition and Culture* 5/1-2: 165-190.

Sorin Costreie is associate professor of philosophy at the University of Bucharest. His main research domains are philosophy of mathematics, philosophy of language, early analytic philosophy (especially Frege), and early modern philosophy (especially Leibniz).

Academic affiliation: Department of Theoretical Philosophy, University of Bucharest, 204 Splaiul Independenței, 060024, Bucharest, Romania. Email: sorin.costreie@unibuc.ro