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A way to see the interplay between theory and reality with a look at the quantum case

(Una forma de ver la interacción entre la teoría y la realidad con un vistazo al caso cuántico)

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ABSTRACT: The purpose of this paper is to argue that neither mathematics nor logic can be applied 'directly' to reality, but to our rational representations (or reconstructions) of it, and this is extended to scientic theories in general. The difference to other approaches (e.g., Nancy Cartwright's, Bueno & Colyvan's or Hughes') is that I call attention to something more than what is involved in such a process, namely, metamathematics. A general schema of 'elaboration' of theories, which I suppose cope with most of them, is presented and discussed. A case study is outlined, the quantum case, whose anchored description, in my opinion, demands a different metamathematics and a different logic.

KEYWORDS: theory vs. reality, metamathematics, quantum physics.

RESUMEN: *El propósito de este artículo es argumentar que ni las matemáticas ni la lógica pueden aplicarse "directamente" a la realidad, sino a nuestras representaciones (o reconstrucciones) racionales de la realidad, y esto se extiende a las teorías científicas en general. La diferencia con otros enfoques (por ejemplo, el de Nancy Cartwright, el de Bueno & Colyvan o el de Hughes) es que llamo la atención sobre algo más de lo que está involucrado en tal proceso, a saber, la metamatemática. Un esquema general de "elaboración" de teorías, que supongo que se adaptan a la mayoría de ellas, se presenta y discute. Se esboza un estudio de caso, el caso cuántico, cuya descripción afianzada exige, en mi opinión, una metamatemática diferente y una lógica diferente de la clásica.*

PALABRAS CLAVE: teoría y realidad, metamatemáticas, física cuántica.

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The lesson for the truth of fundamental laws is clear:
fundamental laws do not govern objects in reality;
they govern only objects in models.

Nancy Cartwright (1983, p. 13)

1. Introduction

In his book *Ensaio Sobre os Fundamentos da Lógica* ('Essay on the foundations of fogic') (da Costa 1980), which fortunately is getting an English translation,[1] Newton da Costa discusses the relationships between mathematics and reality. To him, mathematics and logic "constitute just one discipline" (*ibid.,* p. 212), so we can say that he is also speaking of the relationships between logic and reality. Since my account on the subject, presented below, agrees with him in several aspects, I start with a little revision of his approach in order to formulate mine.

So, the paper is organised as follows. In the next section, we revise some traits of da Costa's approach in order to motivate ours. Then a 'general schema' for understanding the steps in the elaboration of scientic theories is presented and discussed. Next, the role played by the metamathematics is emphasised. The last part deals with our sample case. In considering a metaphysical view (according to the general schema) according to which quantum entities are seen as non-individuals, we argue that a different logic and mathematics may be required to cope with them. The last section discusses the issue of 'going back to reality'. General conclusions are then advanced.

2. da Costa on the application of mathematics to reality

Da Costa distinguishes between 'direct' and 'indirect' applications of maths to reality. In the direct applications, he says, the common objects to which we are making the application behave as they obey the mathematical laws. He exemplies the case where we 'sum' two men plus two men to form four men, which according to him is a 'direct' application of the arithmetical rule '2 + 2 = 4'. Thus he points out that

> [Such a fact] seems obvious that in what respects the simple arithmetic properties (and also the geometric ones) of the concrete objects.
>
> Under certain aspects, with such objects and their inter-relations, one can discern certain logico-mathematical structures with which we are accustomed. All happens as if objects *participate* of the platonic structures of the formal sciences. (*ibid.,* p. 214)

I would like to make a remark about such a proposal, which I will delineate in more detail below. Really, I believe that we do not apply maths or logic *directly* to the world and, by ex-

[1] The translation is being done by Luis F. Bartolo Alegre, from the Un. Mayor de San Marcos, Peru. Good reviews of this book are Rolando Chuaqui's (1991) and Graham Priest's (2000), the last one referring to the French version (da Costa 1997).

tension, the same applies to physical theories (to which I shall be restricted). First of all, in the example cited, there is no such thing as an operation of 'addition of men'; furthermore, as P. Suppes (1998) says, "We cannot literally take a number in our hands and 'apply' it to a physical object". Suppes continues: "What we can show is that the structure of a set of phenomena under certain empirical operations and relations is the same as [that is, is isomorphic to] the structure of some set of numbers under corresponding arithmetical operations and relations." That is, we need to work with *structured* phenomena, with representations of them.

What we do is to apply maths to our (mathematical) *representations* (or *models*) of some parcel of reality and get 'conclusions' about some things.[2] In analysing a domain, we 'reconstruct' it within (in general) our preferred (or learned) mathematics and work with such a 'model'. As we shall comment below, the mathematics we use to do that has also its importance in some discussions.

When da Costa says that the objects in our empirical domain behave as if they obey mathematical rules, I suppose this is what he means: *representation*, turning the observed men into a mathematical element (a set, say) to which the mathematical rules can be applied. Similarly, in physics, we must acknowledge that the laws of physics refer to idealised entities presented in our theories (Cartwright 1983).

It would be interesting to have at least an opinion about this 'process' of *creation* of a scientific theory in order to further discuss it. Furthermore, I believe that logic and mathematics are among our creations, as well as scientific theories (such as those in physics or biology). In a certain sense, they are equally empirical; see Bacciagaluppi (2007) for a discussion on the empirical feature of logic and further references. Below there is a general schema that I hope to capture at least a portion of this process of theory formation.

Let us insist that in order to apply the above-mentioned arithmetical rule to men, we need to make a lot of assumptions, such as that the four men are pairwise distinct, among other things. That is, we need to make suppositions in order to assert that some mathematical rule is being applied to a 'real' domain since we cannot just to 'look' at them and think that we are using our maths 'directly' to count a group of men. Anyway, da Costa later says that "there are no pure direct applications: they are always idealisations" (*ibid.,* p. 215). This additional remark clarifies what he said before and conforms to what we will say in the sequence.

But the more interesting part concerns the indirect applications, which we report as the only ones that exist. He says that while in the direct applications of a mathematical theory T to the 'reality', the real situation constitutes a *model* of the mathematical theory,[3] in the indirect applications we *substitute* a concrete situation (he calls it 'S') by a mathematical theory T, so that T becomes a model of S; again he qualifies by saying that this terminology is ambiguous.[4]

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[2] Bueno & Colyvan (2011), so as Hughes (1997), speak of 'deductions' within the model. I am not sure that the only way the scientist makes inferences is by deductions. Maybe she uses more general ways of inferring in order to get her 'conclusions' which will later be confronted with experiments.

[3] He advises us that this is said without much rigour, and italicises the word 'model' to show the generic use of this word.

[4] The reader can be sure that we are aware of the discussion on the formalisation of a pre-theory, if in a first-order or in a higher-order language; Joseph Melia (1995) brings an account of this discussion which does not concern us here.

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It is important to be enlightened here: the word 'model' has two meanings, and I shall diferentiate them in my schema below. But just to anticipate, one is used when we say that T 'is a model' of S, but perhaps it would be preferred to say that T is a 'mathematical representation' of S in the sense used by the applied mathematician or the engineer, as when one 'represents' the predator-prey situation by a system of differential equations. That is, we have a mathematical 'theory' coping with some aspects of S, but let me emphasise that this will depend on the way we 'understand' the situation, that is, it depends on the scientist's skills and abilities, summing up, of her 'phenomenology'. The other use is in the sense that *S models* the theory T in some way, which seems to induce a 'logical' model, in the sense of mathematical structures that 'satisfy' the axioms of the theory (hence we need to have axioms).[5]

The possibility of applicability of maths to reality is possible, according to da Costa, since "this one is constituted in such a way that, in our relationships with it, there are certain invariants that can be 'captured' by mathematical structures." (*op. cit.,* p. 214). I would also argue that we don't know what reality is. We will never be able to fully understand its intricacies and we have only a phenomenological account of it. So, these invariants are given by *the way we conceptualise* the world, and not from the world itself.

It is precisely at these points that I think several things must be considered, as we shall see below.

3. *A general schema*

The schema I shall present in this section has an interplay with both Hughes' (1997) and Bueno & Colyvan's (2011) schemes, but I think it is more embracing and introduces a discussion which is missing in these mentioned authors, namely, the role played by the metamathematics in the formation of the theory and its models.

In the schema of scientific representation (we use this term in the sense of Suppes (2002),[6] we should distinguish between the things in themselves (*Dinge an sich*) and their mathematical representations in a mathematical (usually, a set-theoretical) structure. We can give a rough idea using the figure 2, and here we have the explanations.[7] The 'quantum case' will be mentioned below.

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[5] Some authors confound (in my opinion) the first writings about the so-called 'semantic approach' to scientific theories. For instance, Elisabeth Lloyd, recalling van Fraassen's 1980 book, quote him in saying that "the essential job of a scientific theory is to provide us with a family of models, to be used for the representation of empirical phenomena" (van Fraassen 1980, p. 310). But then she adds that "[a] theory can be characterized more or less formally [that is, by its models], without first defining a set of theorems" (Lloyd 1994, p. 15). No, we cannot have 'models' without something that collect them in some way. This task is played by the axioms of the theory (the 'theorems' in her words).

[6] Although Suppes discusses the details of such a concept, to us it suffices to agree with him that "[a] *representation* of something is an image, model, or reproducing of that thing." (p. 51). In our case, it means the way we found to mathematically describe a concept in the considered structure which copes with the notion or thing we are interested in (we shall leave out other forms of 'models', such as iconic models).

[7] This schema was presented by me in several places, but here I follow (de Barros, Holik and Krause, forthcoming).

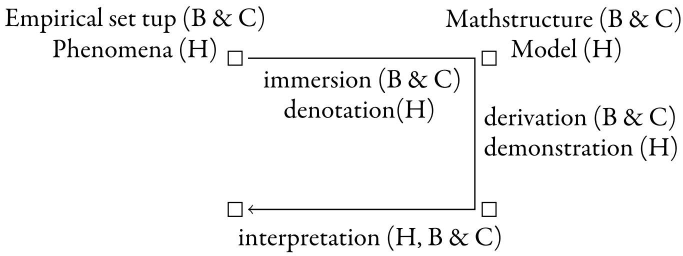

Figure 1

Hughes and Bueno & Colyvan schemes.
Nothing about the metamathematics where these mappings are constructed is considered.
For instance, are the models (math structures) *sets* — even in the quantum case?
Are the demonstrations (derivations) made in classical logic? How are the interpretations defined?

We can admit that the portion of reality Δ we are dealing with is a 'blurry reality', which Bernard d'Espagnat (2006) referred to as *veiled*. According to him, maybe things that are really sharp are just hidden behind a transparent curtain. In my opinion, most of the things in Δ are in fact ontically vague, in particular quantum entities, as we shall see: reality seems to be vague in itself, and not just the language we use to speak about it.[8]

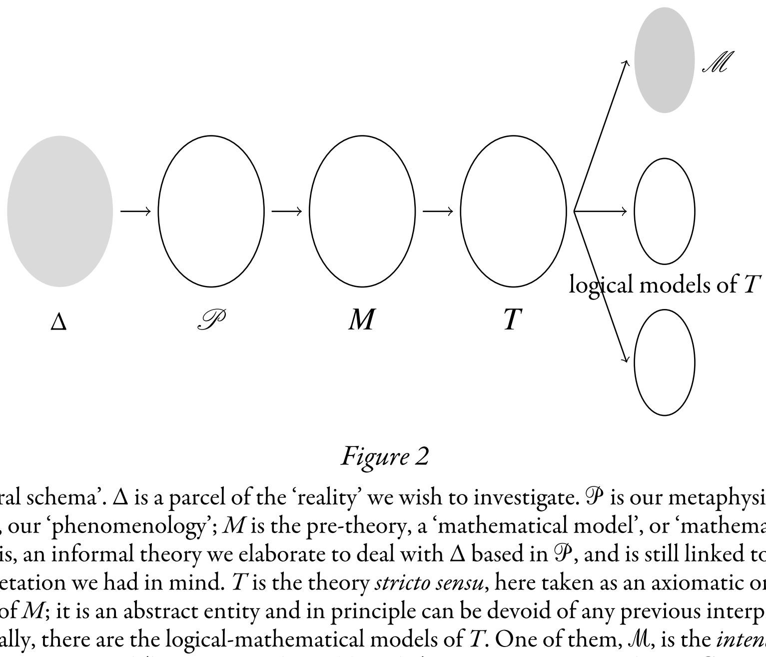

Figure 2

The 'general schema'. Δ is a parcel of the 'reality' we wish to investigate. $\mathscr{P}$ is our metaphysical schema about Δ, our 'phenomenology'; M is the pre-theory, a 'mathematical model', or 'mathematization' of Δ, that is, an informal theory we elaborate to deal with Δ based in $\mathscr{P}$, and is still linked to the initial interpretation we had in mind. T is the theory *stricto sensu*, here taken as an axiomatic or formal version of M; it is an abstract entity and in principle can be devoid of any previous interpretation, and finally, there are the logical-mathematical models of T. One of them, $\mathscr{M}$, is the *intended one*, prepared (by correspondence-like rules) to represent Δ according to $\mathscr{P}$

[8] An updated collection on ontic vagueness is (Akiba and Abasnezhad, 2014).

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In general, philosophers tend to accept that the objects of the world are sharp and well-defined, but that our concepts may be vague (Lewis, 1986; Williamson, 1994), that is, vagueness would be a feature of language.[9] For example, Mary is supposed to be a well-known girl, but the predicate 'intelligent' is not, so there is a kind of vagueness in saying that Mary is intelligent. This suggests that things like girls and other physical objects in our macroscopic scale look vague due to their properties. The case of quantum objects will be discussed later.

A worldview about Δ can be expressed partially, as we sketch out (even unconsciously) a metaphysical view of it through our senses and minds. Here we call it "$\mathscr{P}$'", which reflects our initial views about it. Then, $\mathscr{P}$ congregates what we can call a 'phenomenology', a *Weltanschauung* (von Weizsäcker 2014). For instance, Newton thought that light was made of corpuscles, while Huygens believed in waves, two distinct metaphysics linked to the same phenomenon.

The '$\mathscr{P}$' part of the schema guides us in the elaboration of a phenomenal (or mental) model M to deal with Δ according to our $\mathscr{P}$ which leaves our minds and, of course, can be put in the paper, turning to what Popper could call 'autonomous', shareable, an object of his 'world-3' (Popper 1978).

With M, we formulate concepts and relate them in 'theories', better called *pre-theories* since we have assigned the name *theory* to the next stage. These pre-theories have already the germs of axiomatization, the basic notions and assumptions, which will become the postulates of a later T. For instance, the predator-prey case uses dierential equations firstly proposed in an informal (not axiomatic) way;[10] Cantor's naïve set theory incorporated already the notion of extensionality (which states that sets are equal if they have the same elements) and many others, typical of the theory of sets which were later basic for Zermelo's first axiomatization.

Most applied mathematicians, engineers and physicists (biologists almost for sure) work at this stage, that is, until M of our schema and the results they get are supposed to make reference to Δ, and they *read* Δ through M although, as remarked already, there is much to say about this passage. Darwin's pre-theory of natural selection can perhaps be thought of as an example; another case could be Galilei's theory of the falling bodies, so as the geometry of the ancient Egyptians and Babylonians (previous to the Greeks). Pre-theories are usually only informally stated, formulated using the resources and skills the scientist knows and the resources the scientist knows or even develops, out of formalisation.[11]

Despite the pre-theories can, at first glance, be enough for most of what the standard applied scientist needs, they are not adequate for foundational studies or for the study of the theories themselves, so we need to go further, passing to the axiomatic or even to a formalised version of them. For instance, Newton's mechanics (perhaps in its more precise Lagrangian form, or according to McKinsey, Sugar and Suppes (1953) account), involves

9 For instance, David Lewis says that "The only intelligible account of vagueness locates it in our thought and language." (Lewis 1996, p. 212).

10 The Lotka-Volterra equations, which are used to 'model' the predator-prey case, were posed within the field of Mathematical Analysis, which was axiomatised only later.

11 Enough to recall that Fourier developed his theory of the 'Fourier Analysis' from an analysis of heat transfer.

Galilei's theory of falling bodies, and today there are several alternative formulations of the theory of natural selection.

It is obvious to the reader that from the same domain Δ, different scientists can develop different or even non-compatible pre-theories, depending on their prior knowledge, skills, and preferences. The same happens with the passage from M to T, since in general there is no just one way to axiomatise a pre-theory. The theory T is then taken as representing an axiomatic or even a formalised version of the pre-theory. Euclidean geometry can be taken as a theory of the pre-geometry of the people before Euclid, and if we are to be more rigorous, we could resort to Hilbert's axiomatization of Euclidean geometry (Pogorelov 1987). Other examples come to mind easily.

But an abstract axiomatic or formalised theory T has infinitely many abstract mathematical 'logical' models, usually set-theoretical structures that satisfy its axioms. Even a *categorical* theory, such as the (not elementary) complete field of real numbers, has infinitely many logical models, all of which are isomorphic. The field of real numbers, for instance, has as models the 'reals' given by Dedekind cuts, or by equivalence classes of Cauchy sequences, among other alternatives, all isomorphic, but *different*. These models are not limited to one.

We can select one of them to be our *intended model*, which we endow with a way of representing Δ. Consequently, the terms of the language now refer to organisms, genes, masses, forces, electrons, etc, and sentences are formulated accordingly. Notice that all of this is speculative and made under a hypothesis: in the most complicate cases, we generally don't know several details about Δ, so we leave them out; the phenomenological and metaphysical aspects we get from expertise and experience are also dependent on our skills: we conjecture. The T-version is still more abstract. It can be thought of as something given axiomatically by a set of postulates, usually formulated in the language of set theory, a *set-theoretical predicate*. As described by P. Suppes (2002) (da Costa and Doria, 2022), such a predicate can be satisfied by mathematical structures, which turn to be the *logical models* of the theory described by the predicate. These are abstract mathematical structures, but we can take one of them and provide it with an interpretation in terms of our understanding of Δ and our $\mathcal{P}$, giving 'sense' to its theoretical terms.[12]

This will be our *physical* or *conceptual* model. For instance, we can simulate certain physical systems by harmonic oscillators or the molecules of a gas by a mathematics that mimics billiard balls. One of such models can then be assumed to be our *intended model*, that one that 'describes' the part of Δ we are interested in. This chosen model is 'prepared' to represent Δ (modulo our phenomenology) by means of *correspondence rules* (see the last section below) in the style proposed by the logical empiricists.

I think one of the biggest problems is how we went back from models and theories to Δ, to reinterpret what we were given and discover more about the domain. This will be considered below.

So, the relationships between mathematics, logic, and reality involve not only purely mathematical processes but also informal ones. To put it simply, our theories and models are based on our representations of parcels of reality and not on the actual reality itself (di-

[12] We agree with Dalla Chiara and Toraldo di Francia in that there are no 'empirical' terms: all of them are 'theoretical'. See their (1981) and (Toraldo di Francia, 1981).

rectly). All of this is formulated in certain metamathematics so that we can explore the semantic rules of formal logic in providing the links, something which cannot be achieved if the 'models' are informal. As we will see shortly, the consideration of metamathematics becomes relevant. In short, in order to deal with the world, or with parcels of it, we need to *represent* it first. Usually, we do it by means of mathematical structures.

Despite the reference to 'mathematisation', the same could be said of other not exactly 'mathematical' theories. In the study of the human brain, for instance, one uses concepts such as thinking, resemblances, learning, perception and decisions among others, which can also be thought of as forming a *structure* and, in principle at least, could be thought as possible of being axiomatized.[13]

4. Models as mathematical structures

Standard Model Theory deals with first order or order 1 structures only (Button and Walsh, 2018; Chang and Keisler, 1992). Such structures are composed of one or more domains and operations, relations and distinguished elements over these domains. We do not quantify, for instance, over relations whose arguments are also relations or operations over the individuals of the domain. But, in several situations, we need to quantify over subsets of these domains or on relations or operations that relate not only the elements of the domain(s) but other relations and operations over these elements. For example, take a topological space. A topological space is a structure of the following 'species': $\mathcal{T} = \langle D, \tau\rangle$ where D is non-empty and τ is a collection of subsets of D, the *topology* and some well-known axioms must be obeyed by the elements of τ.[14] Thus, we are involved with things that do not relate only to the elements of the domain, but collections of sets of elements of D. This is something the philosopher of science should take into account: most of the scientific (and even mathematical) structures are not order-1 structures, and so cannot be dealt with by standard Model Theory and, let us recall, there is not a 'Model Theory' for higher-order structures, so we simply don't know what holds in the general, needing to examine the particular situations. A typical example is McKinsey, Sugar and Suppes classical particle mechanics, which is a structure of the form

$$\mathfrak{P} = \langle P, T, m, \vec{s}, \vec{f}\rangle, \tag{1}$$

where P is a set of 'particles', T is an interval of time, m is a function representing the mass of the particles, $\vec{s}$ is the position vector and $\vec{f}$ congregates the forces exercised among the particles, all of this subject to suitable axioms (McKinsey, Sugar and Suppes, 1953). Of course, this is not a order-1 structure.

13 We just recall P. Suppes' work on the foundations of psychology, summarized in Batchelder and Wexley's chapter in (Bogdan 1979).

14 The notion of *species of structures* came from Bourbaki (2004). See da Costa and Krause (2020).

5. The metamathematics

How does all of this affects the development of scientific models? Many things indeed. We need to consider also that the abstract, 'logical', models are mathematical structures, so they are erected in *some* mathematics, generally a set theory. That is, when we represent the physical entities, which exist in Δ according to our phenomenology (the '$\mathcal{P}$' part in the previous schema), we 'conceptualise' them putting them in some informal model, schematised as a pre-theory and in most cases we use set-theoretical devices. But if the objects are quantum entities and it is assumed (this is a metaphysical hypothesis) that they are non-individuals (Krause, Arenhart and Bueno 2022), then the selected set theory becomes much more relevant. Really, we are accustomed to reason in terms of individuals; classical logic, standard mathematics (mainly geometry) and even classical physics were built with the idea of individuals in our minds.[15] So, if our phenomenology requires that this conception is to be changed, it is reasonable to expect that the metamathematical basis changes accordingly.

The importance of paying attention to the metamathematics we use to build the pre-histories and the models of 'T' is noticed by a few people. Really, for the elaboration of the theories in the 'T' aspect of the previous schema, it might be relevant to consider other situations involving the quantum case, which requires attention to the mathematical aspects.

In order to give a few examples, let us consider the following hypothetical situations. Unfortunately, they require from the reader some knowledge of logic and mathematics which cannot be revised here. In the Hilbert space approach, one usually makes use of unbounded operators over the relevant Hilbert spaces, such as those that stand for position, momentum or energy.[16] So, the metamathematics need to be able to accept their existence. But what happens if instead of a standard set theory (such as the ZFC system) or even the quasi-set theory mentioned below, we use the so-called Solovay's set theory (or Solovay's 'model'), which is ZF (ZFC without the axiom of choice) plus DC, the Axiom of Dependent Choice (that is, Sol = ZF + DC)? In such a theory, every linear operator over a Hilbert space is bounded (Maitland-Wright, 1973); we would be in trouble for using the above formalism.

The same would happen if instead of a standard set theory such as the ZFC system, we make use of ZFA, the Zermelo-Fraenkel system with atoms, entities that are not sets, but which can be elements of sets (Suppes, 1972). The problem is that we can construct 'permutation models' of ZFA such as those of Hans Läuchli, which enable the construction of Hilbert spaces with no basis or with bases of dierent cardinalities (Jech 1977). Since the existence of bases is fundamental for the H-space formalism, we would be in trouble.

In 1976, Paul Benioff published two papers (1976, 1976a) where he shows that not every model of ZFC can be used to construct quantum mechanics (that is, a model of it). The details are not relevant here, but the result is that we need to know where we are working, that is, which metamathematics we can use.

[15] Roughly speaking, an individual is something that (i) is *one* of a kind, (ii) presents identity conditions, and (iii) can be re-identified as such in different contexts. For details, see de Barros, Holik, and Krause (forthcoming); Bueno (2023).

[16] A *bounded* operator T is a linear operator over the Hilbert space so that there exists a natural number N such that for all vectors α, we have that $||T(\alpha)|| \leq N||\alpha||$. If T is not bounded, it is unbounded.

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Now an example involving set theory. Think of a set theory such as the ZFC system, axiomatised as a first-order theory. If consistent, it has models but is not categorical, that is, its models are not isomorphic; for instance, due to the Löwenheim-Skolem theorems, it has not only infinite models of any infinite cardinality but *also* a denumerable model. The question is this: Where do these models come from? Notice that while the structures presented in the previous section are *sets* of, say, ZFC or of quasi-set theory, the models of ZFC cannot be sets of ZFC (supposed consistent). This is prohibited by the Second Incompleteness Theorem (Smith 2021, Chap. 17).

As a result, a theory that meets certain conditions of recursivity, expressiveness, and consistency cannot construct its own model. The models of a theory like ZFC need to be considered in strong theories such the KM system (Kelley-Morse set theory) or those involving *universes* or (equivalently), assuming the existence of inaccessible cardinals (see Roitman 2013; Jech 2003).

All of this shows that we should be careful when considering the mathematics that can be used to construct any theory of sets since it is also a scientific theory. As mentioned earlier, the quantum case is our sample case. Let us consider it now.

6. *Exploring the quantum case*

Thus we have a problem, which can be summarised as follows: if collections of quantum objects are not sets in a standard sense since their elements may be indistinguishable, while a standard set is (as put by Cantor) a collection of *distinct* things, we have a foundational problem yet it is acknowledged that the quantum formalism, whatever it is (Hilbert spaces, path integrals, abstract quantum logics, etc.) is well understood. As far as the *interpretations* are concerned, they are the explanations for what is happening. Here we have no space to discuss this topic, so we just mention our own interpretation: following Schrödinger, we consider that the standard theory of the identity of classical logic does not apply to quantum objects.[17]

The second assumption follows Heinz Post (1973); see also (French and Krause 2006) in which the 'non-individuality' of quantum entities should be ascribed not a posteriori as usual, when individuals *are made* to mimic non-individuals within 'standard' frameworks, but their non-individuality is taken "right from the start" (Post's words); in our account, as a primitive concept.[18]

In certain situations, these entities can be considered *absolutely indiscernible* without fulfilling identity conditions. Quantum theory presents lots of examples, such as bosonic condensates and electrons in entangled states. Due to this, it has been acknowledged that collections of such entities should not be taken as elements of a standard *set*, say of ZFC or ZFA; see (Krause 2023) for a discussion and quotations from relevant scientists. Thus,

[17] Really, Schrödinger didn't mention 'classical logic', but just 'identity' and 'sameness'. But we assume that identity is a logical notion and that the *standard* theory is that given by classical logic of first or higher-order or even of a standard set theory such as ZFC. For details, see (Krause 2023; de Barros, Holik, and Krause 2023 forhtcoming; Arenhart 2023; Bueno 2023).

[18] Usually, the notions of 'identity' (or 'sameness') and 'individuality' are confused, but should be discerned; see Arenhart (2023); Bueno (2023); de Barros, Holik, and Krause (forthcoming).

where can we elaborate on the relevant structures that cope with such an assumption of indistinguishability? The approach we suggest is to use the *theory of quasi-sets* where collections of elements (quasi-sets) are characterised by a certain quantity, mediated by a 'quasi-cardinal', but such that no identity condition for the elements can be derived: they can be indistinguishable from each other. The theory enables the attribution of quasi-cardinals to such collections *without* requiring that their elements are discernible, as guessed by some philosophers.[19]

So, it seems reasonable to use the theory of quasi-sets as our metamathematics. The theory is formulated so that it encompasses a 'copy' of ZFA (hence also of ZFC) where all standard mathematical notions can be developed, such as Hilbert spaces, probabilities, and so on. Thus, we can present the following *quasi*-set-theoretical predicate: paraphrasing McKinsey *et al.* (1953), we can say that quantum mechanics is a structure of the type

$$\mathfrak{M} = \langle S, \{\mathcal{H}_i\}, \{\hat{A}_{ij}\}, \{U_{ik}\}, \mathcal{B}(\mathbb{R})\rangle, \quad (i \in I, j \in J, k \in K) \tag{2}$$

where S is a *quasi-set* whose elements stand for the quantum systems we are dealing with, $\{\mathcal{H}_i\}$ is a collection of Hilbert spaces, one for each quantum system in S, the $\hat{A}_{ij}$ are self-adjoint operators defined on $\mathcal{H}_i$ which represent the observables that can be measured according to the quantum rules, and the U_{ik} are unitary operators (also over $\mathcal{H}_i$) that provide the dynamic of the system (Schrödinger's equation), while $\mathcal{B}(\mathbb{R})$ is the set of all Borel sets of the real number line.

The structure is, of course, supplemented by suitable standard postulates (Jammer, 1974; de Barros, Holik, and Krause, forthcoming). The way to speak of entities in such a formalism can be seen when we attribute to each system $s \in S$ a 4-tuple of the form

$$\sigma = \langle \mathbb{E}^4, \psi(\mathbf{x}, t), \Delta, \mathfrak{p}\rangle, \tag{3}$$

where $\mathbb{E}^4$ is the 4-dimensional *Euclidean* space and $\psi(\mathbf{x}, t)$ is the wave function (state-vector), with $\mathbf{x} \in \mathbb{R}^3$ and t ranging over an interval of the real numbers taken as representing instants of time.[20]

Δ is a Borelian (in the real number line) and $\mathfrak{p}$ is a function, defined for some i, determined by the physical system s, in $\mathcal{H}_i \times \{\hat{A}_{ij}\} \times \mathcal{B}(\mathbb{R})$ and assuming values in $[0, 1]$, standing for the probability that a measurement of an observable A, represented by a self-adjoint operator $\hat{A} \in \{\hat{A}_{ij}\}$ lies in the Borelian $\Delta \in \mathcal{B}(\mathbb{R})$. The Hilbert space $\mathcal{H}_i$ is the space of the states of the system considered. The postulates that describe the behaviour of p are those of Mackey (1963, pp. 62, ff.). Notice that for the one-particle system, the configuration space is the ordinary space $\mathbb{R}^3$.

If our metamathematics is quasi-set theory, we need to make clear things such as the way to attribute Hilbert spaces to indistinguishable quantum entities. We cannot discuss this point in detail here, but the reader can believe that the theory generalises the notion of

[19] In fact, some philosopher guess that when we attribute a cardinal to a collection, we are also attributing them an ordinal, hence making the elements discernible one each other (see Krause, 2023). But this conclusion can be overcome in quasi-set theory, as shown in (Krause and Wajch, 2023).

[20] For a generalization to n systems, see (Prugovecki, 1981, p. 120).

function to *quasi-functions* and these entities are used to provide the attribution in a *quasi-set semantics* (da Costa and Krause 1997; French and Krause 2006).

Summing up, in order to build our quantum theory, we suppose that there exist an empirical domain comprising physical quantum entities and formulate a metaphysics about them; in our case, they are taken to be non-individuals. But if you adopt Bohmian quantum mechanics, our phenomenology will say that they are individuals endowed with identity, even if they are 'identical' quanta (Tumulka 2022, §6.1.4).[21] Assuming that a pre-theory is formulated, say any pre-axiomatised version such as Heisenberg's or Schrödinger's pictures. To continue ahead, we can consider an axiomatised theory, such as von Neumann's and then it becomes abstract. The 'logical models', in our case of non-individuals, are better seen in quasi-set theory.

7. *Back to 'reality'*

In my opinion, the more intricate aspect of the above schema is the 'return' from the results got in the model to the empirical domain. If the first part, representation, is buy its own a huge difficult task, the way 'back to reality' is much more rugged. Really, I think that there is no precise (I mean, 'logical') way to associate the results got in the mathematical structures of the theory (M or T) with the elements of reality in Δ (analogous to Bueno & Colyvan's or Hughes' 'interpretation',[22] that is, the route from the mathematical structure to 'the empirical setup' in their schema of 'inferential conception of applied mathematics'). I use this to make an analogy in trying to explain this point: in computation theory, there is a conjecture called 'Church's Thesis' which says that all computable functions can be calculated by an effective method, say by a computer (roughly speaking, by means of recursive functions). Recursive functions have a well-given mathematical definition, but the notion of 'computable function', that is, a function on natural numbers, something that with a certain exaggeration can be said can be performed 'by hand', is informal and quite vague: which functions can be supposed to form part of this class? So, how to mathematically relate these two notions? It is impossible to do it within a mathematical system such as ZFC since computable functions are not perfectly characterised. We need to *assume* Church's thesis or to reject it; it cannot be proven.

The same happens with the elements of the model $\mathcal{M}$ we have chosen as our intended model for the elements in Δ. We need to *postulate* that insofar as the experiments corroborate the model, that is, insofar as the results save the appearances or are empirically adequate, we have the reasons to accept the theory and the chosen $\mathcal{M}$ (van Fraassen 1980). In metamathematics, a domain we enter when we construct abstract theories and models of empirical domains, there is no 'direct' association with reality, so we can grant our be-

[21] In fact, in dealing with 'identical' quanta —indistinguishable in our language— Bohmian mechanics starts attributing labels to the entities (particles), which distinguish them. But after moving to a configuration space, some rules of invariance by permutations are assumed so that the indiscernibility of the elements is 'made by hand'. See the mentioned reference.

[22] Bueno and Colyvan, as well as Hughes, use the same term 'interpretation' to mean the passage from the mathematical results got in the model to the empirical reality.

lief in our theory/model due to other criteria, such as consistency (Hilbert) or nontriviality (da Costa).[23]

The relationship between theory and 'reality' is something we construct and depends on the ways we conceptualise the domains we are interested in. Usually, it is an informal step, and kinds of 'correspondence rules' (Carnap 1966, Chap. 24),[24] are postulated as the old logical empiricists, to provide a way of picking out a class of physical things that correspond to the terms of the theory (Halvorson 2016); these are "the set of rules [that] provide a means for defining theoretical terms" (Carnap, *ibid.,* p. 234).

Supposedly, in relating Δ with $\mathcal{M}$ we need also to go from $\mathcal{M}$ to Δ, so we are required to consider the inverse image of these correspondence rules; thus, if after representing my left hand's fingers by a set $F = \{a, b, c, d, e\}$ and proving that there is a bijection from this set to the von Neumann ordinal $5 = \{0, 1, 2, 3, 4\}$, we can go back to my hand and say that it has five fingers. But this correspondence is informal and something we need to *postulate*; we cannot 'prove' it exists. But this brings a problem. As recalled by Hughes *(op. cit.),* the idea that the theory (and its models) can be seen as a *map* of the territory represented by Δ is not completely feasible. A map, says Hughes, refers to (supposedly) existing things, while the theory (and the models) deal not only with the actual, but also with idealised and merely possible entities. Much care is necessary in establishing the link from the model $\mathcal{M}$ (or of the theory M) to Δ.

8. *Ending by now*

We cannot go into details here, so I suggest the above references for more. But it seems clear that:

1. In the logical analysis of scientific theories, we ought to pay attention to the metamathematics we use. Today we are aware that there are plenty of different 'mathematics' and the choice of one of them is made by pragmatic criteria, such as simplicity, efficiency and even preferences. However, the metamathematics would not be used unconsciously, since in some cases, the explanation of the used metamathematics may be necessary for understanding.
2. Really, we need to have studied where the considered notions can be proven to exist; do we need great cardinals? Then we need something stronger than the ZFC system. Do we use notions from category theory? Do we assume an ontology of quantum non-individuals? Then suitable metamathematics need to be carefully chosen.
3. We need also to consider semantic issues, such as considering notions such as truth, satisfaction and logical validity. Again we need metamathematics; as

[23] It is well known that Hilbert has advanced a 'criterion of existence' of mathematical theories, grounded on their consistency. Newton da Costa, assuming paraconsistent logics, says that certain inconsistencies can be admitted but the theory should avoid being trivial, that is, all formulas become theorems. While for Hilbert to exist is to be consistent, for da Costa to exist is to be non-trivial; see (da Costa, Krause, and Bueno 2007).

[24] Other people proposed alternative names, such as Bridgman's 'operational rules' or Campbell's 'dictionary', a term Carnap reputes as feasible to the idea.

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shown by Dalla Chiara and Toraldo di Francia (1993), in the quantum case, the semantical notions should be developed not in something like ZFC, but in a 'set' theory that copes with indistinguishability, and quasi-set theory fulfills the needs (so as their theory of *quasets*, which cannot be confounded with quasi-sets – for a comparison, see Dalla Chiara, Giuntini, and Krause 1998; French and Krause 2006).

4. All of this shows that Suppes is right in saying that we cannot take a number, say 5, and attribute it directly to my hand. We need to 'transform' the fingers of my hand into elements of a set in order to use mathematics.[25]
5. We cannot have a precise (right) account of the reality we are investigating. This is obviously a thesis, but the opposite is equally true. In other words, we cannot be free from metaphysics, yet some such as Otávio Bueno (again) have arguments showing that in considering very weak assumptions such as first-order identity, we can be free of metaphysics; see his (2023). This is a fascinating point that deserves further discussion.
6. We usually make simplifications and construct idealised models to deal with our portion of reality grounded on our metaphysical assumptions, and these models work as far as they 'save the appearances' (van Fraassen 1980) and work, being feasible as far as we feel they work. After this, we simply change them.
7. Even our metaphysical conceptions change. A well-documented case is that of the 'revision' in metaphysics given by quantum theory; as Michel Bitbol has pointed out, "the case of quantum mechanics might well require from us a complete redefinition of the nature and task of metaphysics" (Bitbol 2008).

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[25] Alternative stuff is also possible. For instance, Otávio Bueno defends the idea of avoiding the use of sets and using second-order logic instead; see (Bueno 2023, Bueno & Colyvan 2011).

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