JOURNAL PRE-PROOF

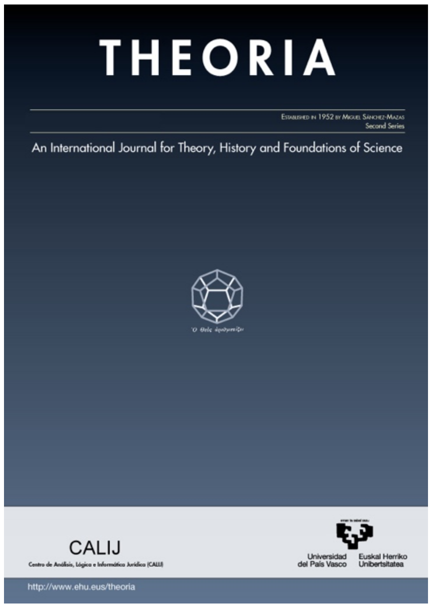
How we learned to stop worrying and love *tonk*

Luis Estrada-González
& Christian Romero-Rodríguez

DOI: 10.1387/theoria.26626

Received: 19/06/2024

Final version: 08/07/2025



HOW WE LEARNED TO STOP WORRYING AND LOVE *TONK*

(Cómo aprendimos a dejar de preocuparnos y a amar a tonk*)*

Luis Estrada-González
Institute for Philosophical Research, UNAM

Christian Romero-Rodríguez*
Institute for Philosophical Research, UNAM

ABSTRACT: According to common wisdom, the connective *tonk* defined by Prior trivializes any theory that contains it. However, it should not be forgotten that whether an argument holds or not depends to a large extent on the underlying notion of logical consequence. Logical consequence is usually assumed to be Tarskian, that is, reflexive, transitive and monotonic. However, Belnap had already conjectured that *tonk* might not be so problematic in a non-transitive logic, which Cook finally proved in 2005. In this paper we improve on Cook's result in two ways: our hypothesis is simpler (namely, we use fewer interpretations than he did and we do not rely on a disjunctive consequence relation) and it is not *ad hoc* (namely, our working consequence relation is not designed merely to avoid triviality under *tonk*).
Keywords: *tonk*, Dunn semantics, non-Tarskian logical consequence, non-transitivity

RESUMEN: Según la sabiduría popular, la conectiva *tonk* definida por Prior trivializa cualquier teoría que la incluya. Sin embargo, no hay que olvidar que, si un argumento es lógicamente válido o no, depende en buena medida de la noción de consecuencia lógica subyacente. Casi siempre se asume que la consecuencia lógica es tarskiana, esto es, que es reflexiva, transitiva y monotónica. Sin embargo, Belnap había conjeturado que *tonk* podría no ser tan problemática en una lógica no transitiva, cosa que finalmente probó Cook en 2005. En este artículo mejoramos el resultado de Cook en dos aspectos: nuestra hipótesis es más simple (a saber, usamos menos interpretaciones que él y no usamos una relación de consecuencia lógica disyuntiva en el *definiendum*) y no es *ad hoc* (a saber, nuestra relación de consecuencia lógica no está definida exclusivamente para evitar la trivialidad).
Palabras clave: tonk, semántica de Dunn, consecuencia lógica no tarskiana, no transitividad

Short summary: Belnap highlighted the role of Transitivity in Prior's triviality proof involving *tonk*, but a non-trivial, non-transitive logic with *tonk* was never developed until Cook's proposal with four interpretations and a disjunctive consequence relation. We improve on that proposal: we show that only three interpretations suffice and that a non-disjunctive consequence relation is not required.

* **Correspondence to:** Christian Romero Rodríguez. Institute for Philosophical Research, National Autonomous University of Mexico, Circuito Mario de la Cueva s/n, Ciudad Universitaria, Ciudad de México, 04510, México. – Christian.romero@outlook.com – https://orcid.org/0000-0002-7211-3392

Introduction

In *The runabout inference ticket*, Prior (1960) introduced the binary connective *tonk*. Prior defines *tonk* by stipulating its introduction and elimination rules. The problem with *tonk* is that its incorporation into certain theories can trivialize them. It is enough to have *tonk* and a transitive logical consequence relation for it to be possible to conclude any formula.

In *Tonk, Plonk and Plink*, Belnap (1962) noted that the introduction and elimination rules of *tonk* could be compatible with other types of logical consequence relations. That is, the trivialising effects attributed to *tonk* should only be restricted to transitive logical consequence relations. Unfortunately, Belnap never provided a non-transitive logic for this purpose. Many years later, in *What's wrong with Tonk(?)*, Cook (2005) introduced a non-transitive logic for *tonk*, which he called *Tonk Logic* (**TL**). Cook takes the language of *First Degree Entailment* (**FDE**) as a starting point, expands it to include *tonk* and defines a new logical consequence relation. In the non-transitive logic presented by Cook, *tonk* is just another connective in the language and does not trivialize.

However, despite Cook's best efforts, his proposal seems contrived for two reasons. The first is that it requires four admissible interpretations to express *tonk*; the second is that the notion of logical consequence used by him is disjunctive and does not seem to be a minimal modification of the Tarskian notion of logical consequence, so it requires too much conceptual motivation. Here we present a proposal that does not have these problems: we show that three admissible interpretations are sufficient to express *tonk*, and we use a notion of logical consequence that is not disjunctive and is more natural than Cook's in the sense that, when used in contexts with only two admissible interpretations, it is coextensive with the Tarskian notion of logical consequence.

The plan for this paper is as follows. In the first section, we introduce *tonk* and the assumptions necessary for its trivialisation. We also present Barceló's (2008) method for converting (bivalent) truth tables into derivation rules (and vice versa). This method is useful for us to present the evaluation conditions of *tonk*. In the second section, we present Cook's **TL** logic and two objections to his proposal. In the third section, we show that there is a way in which *tonk* can validate Prior's rules without appealing to a logical consequence relation, which may seem artificial. Finally, we present a method for identifying what kind of connective *tonk* is, to determine that it is both a conjunction and a disjunction.

1. *The bomb*

Prior (1960) argued that a meaningful connective cannot be specified simply by its introduction and elimination rules. To illustrate his thesis, he proposed a binary connective called '*tonk*', represented here by '$\not\$$', with the following introduction and elimination rules, where A and B are any formulas of the language:

tonk Introduction (I$\nlessgtr$)

$A \vdash A \nlessgtr B$

$B \vdash A \nlessgtr B$

tonk Elimination (E$\nlessgtr$)

$A \nlessgtr B \vdash A$

$A \nlessgtr B \vdash B$

The problem with *tonk* is that it seems to trivialize any theory **T** in which it appears. Consider the following proof:

1. $A \vdash_{\mathbf{T}} A \nlessgtr B$	I$\nlessgtr$
2. $A \nlessgtr B \vdash_{\mathbf{T}} B$	E$\nlessgtr$
3. $A \vdash_{\mathbf{T}} B$	1, 2 Transitivity of $\vdash_{\mathbf{T}}$

The conclusion expresses that, if a theory **T** includes *tonk*, any formula of the language implies any other.

For many, including Prior himself, the lesson of *tonk*'s case is that for a connective to be meaningful it must correspond to a pre-theoretical meaning which cannot simply be captured in a system of rules. Some, like Stevenson (1961) in *Roundabout the runabout inference-ticket*, argue that such a pre-theoretical meaning can be captured, at least to a satisfactory extent, by evaluation conditions such as those underlying truth tables. If this is so, then to show that there can be no pre-theoretical meaning associated with *tonk*, it would suffice to show that *tonk* cannot be evaluated, that it has conditions that cannot be satisfied. Stevenson shows that *tonk* has classically impossible conditions to satisfy, but here we want to present an alternative argument which is clearer to us.

Axel Barceló (2008) developed a method for converting two-valued truth tables into natural deduction rules (NDR) with multiple-conclusions[1] and vice versa, which is particularly useful for showing the peculiarities of the evaluation conditions required by *tonk*. Let us first note that in a multiple-conclusion framework, both rules for *tonk* can be simplified:

Tonk Introduction (I$\nlessgtr$)	*Tonk* Elimination (E$\nlessgtr$)
$A, B \vdash A \nlessgtr B$	$A \nlessgtr B \vdash A, B$

We now explain Barceló's method. Let ©($A_1, \ldots, A_n$) be a formula whose main connective is the n-ary connective ©. Then

I. ©($A_1, \ldots, A_n$) is a conclusion in an NDR if and only if it is true.

II. ©($A_1, \ldots, A_n$) is a premise in an NDR if and only if it is false.

1 For more on multiple-conclusions, see Shoesmith and Smiley (1978).

III. A_k is a premise in an NDR if and only if it is true.

IV. A_k is a conclusion in an NDR if and only if it is false.

Where $1 \leq k \leq n$.

For example, consider the truth table of the extensional conditional (→), where truth and falsity are represented by 1 and 0.

A	B	$A \rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

From each line of the table, we can obtain an NDR. In descending order, the four rules are as follows:

1. $A, B \vdash_{\mathbf{L}} A \rightarrow B$
2. $A, A \rightarrow B \vdash_{\mathbf{L}} B$
3. $B \vdash_{\mathbf{L}} A \rightarrow B, A$
4. $\vdash_{\mathbf{L}} A \rightarrow B, A, B$

In the first case, by (III), A and B are premises because they are both true. By (I), $A \rightarrow B$ is a conclusion because it is true. In the second case, by (III), A is a premise because it is true. By (II), $A \rightarrow B$ is a premise because it is false. By (IV), B is a conclusion because it is false. The same can be done with 3 and 4.

The evaluation conditions obtained from Prior's rules for *tonk*, using Barceló's method, are as follows:

$A \nlessgtr B$ is true if and only if A is true.

$A \nlessgtr B$ is true if and only if B is true

$A \nlessgtr B$ is false if and only if A is false

$A \nlessgtr B$ is false if and only if B is false

Or, rewriting:

$A \nlessgtr B$ is true if and only if A is true or B is true.

$A \nlessgtr B$ is false if and only if A is false or B is false

Here we can clearly see that this connective is undefinable in classical logic, since it requires interpretations that are not admissible in a usual two-valued semantics. Suppose A is true and B is false: according to the evaluation conditions just given, $A \nlessgtr B$ is true (since A is true) and false (since B is false). The same result is obtained by assuming that A is false and B is true. The table for *tonk* would look like this:

$A \not\lessgtr B$	1	0
1	1	1,0
0	1,0	0

One might ask whether this is not a semantics-dependent rather than a logic-dependent result; in other words, one might ask whether *tonk* is not definable in some other semantics for classical logic. The short answer is 'no': there is no semantics S which has the following characteristics:

- S is functionally complete using a classical metatheory.
- A connective with homophonic evaluation conditions for *tonk* is definable in S.
- The Prior rules for *tonk* are valid under S.
- The notion of logical validity is Tarskian.

So far, the bomb: *tonk* trivializes any theory with a semantics S that has the conditions just stated.

In *On three-valued presentations of classical logic*, Da Ré, Szmuc, Chemla and Egré (2023) introduced some logics, with three-valued semantics, which could be considered as presentations of classical logic. However, some of these logics are not reflexive or transitive. Moreover, some of these semantics are not functionally complete. Therefore, even if connectives such as *tonk* are definable in some of these semantics (such as the *tonk* we will present in the next section), they do not satisfy the requirement of being Tarskian or of being functionally complete enough to constitute a counterexample to what has been said above.[2]

2. *How we stopped worrying*

As we have seen, $A \not\lessgtr B$ is true and false if one of its components is true and the other is false. For *tonk* to be expressible, then, at least three admissible interpretations are needed for any formula A: A is (only) true, A is (only) false, and A is both true and false. Fortunately for us,

---

[2] In *Formalization of Logic*, Carnap (1943) presented non-normal semantics for classical logic. In these semantics, interpretations are considered non-normal because, according to Church (1943, p.493), they "contravene in some way the usual interpretation of classical truth tables". Some ways of "contravening" the usual semantics are to increase the number of values or the number of interpretations, or to force non-equivalence between true (respectively false) and non-false (respectively non-true). For example, in Carnap's non-normal semantics, in the evaluation conditions of negation (expressed here with 'N'), it is either the case that, i) NA is true if and only if A is true, or ii) NA is false if and only if A is not false. Tables compatible with these non-normal semantics can also be found in Church (1953). However, these semantics are not functionally complete either, so they do not provide a counterexample to the above. Some criticisms and comments on Carnap's proposal can be found in the aforementioned Church (1944).

and for *tonk*, such semantics are better known and more workable than they were in Stevenson's time.

Before presenting our proposal, we present Cook's (2005) proposal. To do this, it is necessary to first introduce Dunn semantics. Dunn semantics allows us to relate, not necessarily in a functional way, propositional variables to only two truth values, namely, truth and falsity. See more in Dunn (1976). Despite the two-valued nature of the semantics, we can have different sets of admissible interpretations to represent different logics. Omori and Sano (2015) proposed a general method to transform truth tables with up to four admissible interpretations into truth and falsity conditions. In the following, we present Cook's Tonk Logic (**TL**), which uses a Dunn two-valued semantics.

Let L be a formal language for **TL** with a set of formulas constructed in the usual way, from a set of propositional variables VAR = $\{p_1, \ldots, p_n\}$ with the connectives $\sim$, $\wedge$, $\vee$ and $\otimes$, in which '$\otimes$' is the symbol Cook uses for *tonk*.[3] We will use the letters 'A', 'B', 'C',..., of the Latin alphabet as arbitrary formulas of L, and the letters 'Γ', 'Δ',..., of the Greek alphabet as sets of formulas.

An interpretation '*i*' for L is a relation between atomic formulas and the truth values (1 and 0), such that we have the following ways:[4]

- p_i is true but not false, represented by '$1 \in i(p_i)$ and $0 \notin i(p_i)$'; more briefly, $i(p_i)=\{1\}$;
- p_i i is true but also false, represented by '$1 \in i(p_i)$ and $0 \in i(p_i)$'; more briefly, $i(p_i)=\{1, 0\}$;
- p_i is neither true nor false, represented by '$1 \notin i(p_i)$ and $0 \notin i(p_i)$'; more briefly, $i(p_i)=\{\ \}$;
- p_i is false but not true, represented by '$0 \in i(p_i)$ and $1 \notin i(p_i)$'; more briefly, $i(p_i)=\{0\}$.

The interpretations extend to evaluations for all formulas according to the following evaluation conditions[5]:

$1 \in i(\sim A)$ iff $0 \in i(A)$

$0 \in i(\sim A)$ iff $1 \in i(A)$

$1 \in i(A \wedge B)$ iff $1 \in i(A)$ and $1 \in i(B)$

$0 \in i(A \wedge B)$ iff $0 \in i(A)$ or $0 \in i(B)$

---

[3] There is a conditional $A \rightarrow B$ which is definable as $\sim A \vee B$; and a biconditional $A \leftrightarrow B$ which is definable as $(A \rightarrow B) \wedge (B \rightarrow A)$.

[4] We have chosen to work with a Dunn-style bivalent semantics, which is characterized by having only two truth values. Unlike other approaches, the evaluation here is not restricted to a function but is a relation in general. This allows for the consideration of all four possible interpretations, even with only two truth values. The interpretation *i* should not be understood as a function, is used as relation that assigns sets of values to formulas. So the notation $i(p)=\{1\}$ does not presuppose a standard functional interpretation. We are not assuming a classical set theoretic metatheory. Many three-valued and many-valued logics can be presented using a Dunn-style bivalent semantics. Presenting them in this way has significant advantages, as it avoids engagement in debates about the ontological or semantic status of other kinds of values (such as *i*, *b*, *n*, etc.), and allows us to bypass discussions concerning whether such entities genuinely count as truth values. Since this issue goes beyond the scope of the present article, we refer the reader to Estrada-González (2019) and (2022) for a more detailed treatment.

[5] We use the same notation for interpretations and evaluations.

$1\in i(A\vee B)$ iff $1\in i(A)$ or $1\in i(B)$

$0\in i(A\vee B)$ iff $0\in i(A)$ and $0\in i(B)$

$1\in i(A\otimes B)$ iff $1\in i(A)$

$0\in i(A\otimes B)$ iff $0\in i(B)$

The evaluation conditions can be presented in tabular form as follows:

$\sim A$	A
{0}	{1}
{1,0}	{1,0}
{ }	{ }
{1}	{0}

$A\vee B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{1}	{1}	{1}
{1,0}	{1}	{1,0}	{1}	{1,0}
{ }	{1}	{1}	{ }	{ }
{0}	{1}	{1,0}	{ }	{0}

$A\wedge B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{1,0}	{ }	{0}
{1,0}	{1,0}	{1,0}	{0}	{0}
{ }	{ }	{0}	{ }	{0}
{0}	{0}	{0}	{0}	{0}

$A\otimes B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{1,0}	{1}	{1,0}
{1,0}	{1}	{1,0}	{1}	{1,0}
{ }	{ }	{0}	{ }	{0}
{0}	{ }	{0}	{ }	{0}

The logical consequence relation of **TL** is the following: Let A and Γ be a formula and a set of formulas of L, respectively. A is a logical consequence of Γ in **TL**, $\Gamma \vDash_{\mathbf{TL}} A$, if and only if, either, for each interpretation i, if $1\in i(B)$, for all $B\in \Gamma$, $1\in i(A)$; or, for each interpretation i, if $0\in i(A)$, $0\in i(B)$ for some $B\in\Gamma$. According to **TL**'s definition of logical consequence, an argument is logically valid if and only if either: either truth is preserved from premises to conclusion in every interpretation (truth-preserving), or falsity is preserved from conclusion to premises, also in every interpretation (falsity-preserving).

Transitivity is not valid in **TL**:

$$\text{If } A \vDash_{\mathbf{L}} B \text{ and } B \vDash_{\mathbf{L}} C, \text{ then } A \vDash_{\mathbf{L}} C$$

To give a counterexample to Transitivity is to show that $A \vDash_{\mathbf{L}} B$ and $B \vDash_{\mathbf{L}} C$ are valid arguments, while $A \vDash_{\mathbf{L}} C$ is an invalid argument. Specifically, using *tonk*, the following argument is invalid:

$$\text{If } A \vDash_{\mathbf{TL}} A \otimes B \text{ and } A \otimes B \vDash_{\mathbf{TL}} B\text{, then } A \vDash_{\mathbf{TL}} B$$

Using the *tonk* table, we can see that if A is true, then $A \otimes B$ is true, so $A \vDash_{\mathbf{TL}} A \otimes B$ is a valid argument. It can also be seen that if B is false, then $A \otimes B$ is false, so $A \otimes B \vDash_{\mathbf{TL}} B$ is valid. However, $A \vDash_{\mathbf{TL}} B$ is not valid because it preserves neither the truth of the premises to the conclusion nor the falsity of the conclusion to the premises. For example, if $i(A)=\{1\}$ and $i(B)=\{0\}$. A consequence of **TL** being a non-transitive logic is that Prior's triviality proof does not hold for it. Thus Cook ends his attempt to claim *tonk* as a meaningful connective.

However, and despite Cook's best efforts, his proposal seems contrived for two reasons. The first is that it requires four admissible interpretations to express *tonk*; the second is that the notion of logical consequence used by him is disjunctive. We begin by discussing the second reason. It seems to us that defining a disjunctive logical consequence is not a minimal modification of the Tarskian notion of logical consequence because it requires too much conceptual motivation not to seem *ad hoc*. Let us recall the Tarskian notion of logical consequence:

- An argument $\Gamma \vDash_{\mathbf{L}} A$ is *logically valid* if and only if, for each interpretation i, if $1 \in i(B)$, for all $B \in \Gamma$, $1 \in i(A)$.

The definition of logical consequence of **TL** can be obtained from the Tarskian notion of logical consequence by putting its *definiens* in disjunction with that of the following logical consequence relation:

- An argument $\Gamma \vDash_{\mathbf{L}} A$ is *logically valid* if and only if, for each interpretation i, if $0 \in i(A)$, $1 \in i(B)$ for some $B \in \Gamma$.

Cook does not give an argument in favor of his disjunctive logical consequence relation because he considers that it is not too "extravagant or inconceivable" (Cook, 2005, p.221). According to Cook, accepting his consequence relation implies that "there is no reason to prefer the Truth-Preserving Consequence to the Falsity-Preserving Consequence, or vice versa" (Cook, 2005, p.222). Indeed, when classical logic is presented, for example, one can use either the Truth Preservation Consequence or the Falsity Preservation Consequence interchangeably, since both determine the same collections of valid and invalid arguments. From the classical perspective there is thus no reason to prefer one over the other.

There is a reason why we consider the disjunctive logical consequence relation to be an *ad hoc* hypothesis, namely that the preference of the disjunctive logical consequence definition over one that uses only one of the two disjuncts depends entirely on the presence of *tonk* in the language. Say that a disjunctive logical consequence relation **D** is *redundant* if and only if there is a logical consequence relation **N** in whose *definiens* only one of the disjuncts of the *definiens* of **D** is needed, and both **N** and **D** characterize the same arguments (based on a semantics S).

In logics such as **FDE** or classical logic, the disjunctive logical consequence relation characterizes exactly the same arguments as the logical consequence relations based on any of the disjuncts of the disjunctive relation. In **TL**, the logical consequence relation is not redundant, some arguments are valid thanks to one of the disjuncts, but others are valid thanks to the other disjunct (as they would be invalid according to the first one). For example, $A \nlessgtr B \vdash_{\mathbf{T}} B$ is a valid argument with Falsity Preservation, but invalid with Truth Preservation. However, if *tonk* were not part of the language of **TL**, as in **FDE**, the logical consequence relation would be redundant. So, the basis of the disjunctive definition has no justification independent of the possibility of defining *tonk*.

As in other fields, *ad hoc* hypotheses in logic are questionable because they are introduced for the sole purpose of resolving a discrepancy between the theory and a particular result. *Ad hoc* hypotheses, such as the disjunctive logical consequence relation, do not extend the scope of the theory or its ability to solve other logical problems. Here we argue that three admissible interpretations are sufficient to express *tonk*, and we use a notion of logical consequence which is not disjunctive, and therefore its justification is not based on the fact that it allows *tonk* to be defined.

Let us recall Prior's argument:

1. $A \vdash_{\mathbf{T}} A \nlessgtr B$ I$\nlessgtr$

2. $A \nlessgtr B \vdash_{\mathbf{T}} B$ E$\nlessgtr$

3. $A \vdash_{\mathbf{T}} B$ 1, 2 Transitivity of $\vdash_{\mathbf{T}}$

Prior's rules for *tonk* do not in themselves imply triviality, as Belnap (1962) had already pointed out. The conclusion depends crucially on the validity of Transitivity of $\vdash_{\mathbf{T}}$ and not only on Prior's rules. So it seems that there are at least two ways to avoid the triviality result that every formula implies every other formula, either one of Prior's rules must be invalid, or Transitivity must be invalid.

Let us recall the evaluation conditions obtained by Barceló's method:

$A \nlessgtr B$ is true if and only if A is true or B is true.

$A \nlessgtr B$ is false if and only if A is false or B is false

The table for *tonk* would be as follows, using only three admissible interpretations:

$A \nlessgtr B$	{1}	{1, 0}	{0}
{1}	{1}	{1, 0}	{1, 0}
{1,0}	{1, 0}	{1, 0}	{1, 0}
{0}	{1, 0}	{1, 0}	{0}

However, it seems that, in this semantics, the road to non-transitivity is closed, since the notion of Tarskian logical consequence is transitive, and that rather the rule E≸ is invalid. In fact, consider the interpretation in which A is $\{1\}$ and B is $\{0\}$. In this case, the premise of $A≸B \vdash_T B$ is true but the conclusion is not. On the other hand, the way to define Cook's *tonk* using only these three admissible interpretations is also closed: Cook's *tonk* (⊗) is not true and not false even if A is false and B is true. That is, $i(A \otimes B) = \{\ \}$ when $i(A) = \{0\}$ and $i(B) = \{1\}$. This is not the case for ≸. Using the same evaluation conditions of ≸, we can also define *tonk* for semantics with four admissible interpretations:

A≸*B*	{1}	{1,0}	{ }	{0}
{1}	{1}	{1,0}	{1}	{1,0}
{1,0}	{1,0}	{1,0}	{1,0}	{1,0}
{ }	{1}	{1,0}	{ }	{0}
{0}	{1,0}	{1,0}	{0}	{0}

A corollary of this is that Dunn semantics allows us to identify at least two *tonk* connectives. The first, $≸_p$, is defined by Prior-style introduction and elimination rules; the second, $≸_m$, by evaluation conditions.[6] In general, these are different connectives, for the rule E≸ may not hold for $≸_m$, while the connective for which both E≸ and I≸ hold has different evaluation conditions than $≸_m$.

According to Barceló's method, *tonk* is a connective that has the truth condition of a disjunction and the falsity condition of a conjunction. However, ⊗ has neither the truth-condition of a disjunction nor the falsity-condition of a conjunction. ⊗ recovers only part of the evaluation conditions of conjunction and disjunction. For this reason, our *tonk* connective ($≸_m$ according to the notation of the previous paragraph) is not the same connective as Cook's connective (⊗). Cook's ⊗ connective is designed on the basis of the minimal requirements that introduction and elimination rules must fulfil in order to trivialize a logic such as the classical one. For this reason, we consider ⊗ to be more similar to $≸_p$ than to $≸_m$.

Something very similar occurs with the *tonk* defined in Ripley (2015). In that work, *tonk* is introduced through sequent rules, without reference to evaluation conditions. Within that approach, sequents are taken to specify aspects of the meaning of *tonk*. The proposed rules are as follows:

[6] In a different approach, this distinction between *tonks* was also explored by Buacar (2018) and Teijeiro (2020).

$$\textit{tonk}\text{R}\ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \textit{ tonk } B, \Delta} \qquad \textit{tonk}\text{L}\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \textit{ tonk } B \vdash \Delta}$$

To illustrate why *tonk* is problematic, Ripley argues that the Cut rule plays a central role in the derivation of triviality. In doing so, she opens the way for investigations such as ours, which not only aim to diagnose the problem, but also to provide a concrete tool capable of accomplishing the work that Ripley had already outlined. On the other hand, using the method proposed by Béziau (2001, p. 374), it is possible to extract the evaluation conditions of Ripley's *tonk* from its associated sequent rules. These conditions match exactly those of the $\otimes$ connective. Ripley explores the possibility of having her sequent rules validate Prior's original principles. This is why her approach resembles Cook's more closely. Therefore, Ripley's *tonk* is more closely related to $\not\lessgtr_p$ than to $\not\lessgtr_m$. In this sense, our proposal can be seen as extending Ripley's result by exploring alternative ways of modeling a connective like *tonk*.

The question now is whether there is any way in which $\not\lessgtr_m$ can validate Prior's rules E$\not\lessgtr$ and I$\not\lessgtr$, that is, whether we can have it all: a *tonk*-like connective for which Prior's rules are valid and which is described model-theoretically by the truth condition of a disjunction and the falsity condition of a conjunction. The answer is yes, and the simplest way to obtain it is by means of a non-transitive notion of logical validity. Unlike Cook and Ripley's proposals, our approach extracts *tonk* directly from evaluation conditions. We have shown that extraction alone does not ensure the validity of Prior's rules; additional steps are necessary. Ripley and Cook begin oppositely, assuming the validity of the rules from the outset and developing formal mechanisms to avoid triviality. Their versions of *tonk* preserve the truth of Prior's rules within non-trivial frameworks, whereas our proposal does not start from this assumption. Through Barceló's method, we show that a non-transitive context remains necessary for these rules to be valid.

3. *How we learned to love tonk*

One problem with classical logic is that it identifies many properties that perhaps should not be identified. This has been repeated many times for the case of contradiction and triviality, for example, as well as for the pairs falsity/non-truth and truth/non-falsity, which is sufficiently clear in the case of Dunn semantics. Precisely the distinction between falsity and non-truth, on the one hand, and between truth and non-falsity, on the other, makes it possible to distinguish in turn between different notions of logical consequence which are equivalent in classical logic:

tt: The argument $\Gamma \vDash_L \Delta$ is logically valid if and only if, for all interpretation, if the premises are not only false, at least one conclusion is not only false either.

ts: The argument $\Gamma \vDash_L \Delta$ is logically valid if and only if, for all interpretation, if the premises are not only false, at least one conclusion is only true.

st: The argument $\Gamma \vDash_{L} \Delta$ is logically valid if and only if, for all interpretation, if the premises are only true, at least one conclusion is not only false.

ss: The argument $\Gamma \vDash_{L} \Delta$ is logically valid if and only if, for all interpretation, if the premises are only true, at least one conclusion is also true.

The properties that premises and conclusions must fulfil in a logically valid argument are called 'standards' (see more in Chemla et al. 2017, p. 2198). In the logical consequence relations *tt* and *ss*, the standards of premises and conclusions are the same; in *ts* and *st*, they are different, which is why these logical consequence relations are called 'mixed'. For more on *tt*, *ss*, *ts* and *st*, see Cobreros et al. (2012a). For more on mixed logics, see Cobreros et al. (2012b). We have provided the definition of these logical consequence relations, and given the framework of Dunn semantics, they are useful for defining a formal consequence relation when working with three or four interpretations. The only variation lies in the interpretations considered (specifically, what is understood as *tolerant*).

For example, consider the admissible interpretations for **K3**, **LP**, and **FDE**. In these cases, the interpretations to be considered for an argument to be *st*-valid are as follows:

- **K3**: {{1}{ }{0}}
  Standard for premises: {1}
  Standard for conclusion: {1} and { }
- **LP**: {{1}{1,0}{0}}
  Standard for premises: {1}
  Standard for conclusion: {1} and {1,0}
- **FDE**: {{1}{1,0}, { }{0}}
  Standard for premises: {1}
  Standard for conclusion: {1}, { }, and {1,0}

When working with three values, as in some semantic presentations of **K3** and **LP**, it is common to assume that there are different interpretations of the third value. In **K3**, this value is typically interpreted as "neither true nor false", while in **LP**, it is interpreted as "both true and false". Now, when considering mixed consequence relations (such as the *st* and *ts* relations), premises and conclusions may treat this additional value differently, which might lead one to think that two different readings of the same value are being used. As a result, it might seem that there are not just three, but actually four values in play.

However, this objection, which could be extended to claim that four values or interpretations are actually being used, does not apply in our case. The apparent difference in how the intermediate value is treated between premises and conclusion is not due to multiple readings of that value, but rather to the differentiated use of strict and tolerant standards in mixed consequence relations. In other words, even though we are working with three interpretations, there is no third value in our semantics that admits multiple interpretations. Thus, it is clear that there are semantic presentations of the *st* and *ts* consequence relations that, at a conceptual level, do not require assuming two readings of the intermediate value.[7]

In classical logic, the direction in which the standards are connected is also indistinct, i.e. it does not matter whether the connection of standards goes from premises to conclusions or from conclusions to premises. In the four relationships above, the connection is from premises to conclusions. But in contexts in which a distinction is made either between truth and non-falsity or between falsity and non-truth, it is also important to distinguish the direction of connection of standards (see more in Wansing and Shramko, 2008).[8] Thus, we have the following notions of logical consequence:

$\leftarrow$*ff*: The argument $\Gamma \vDash_{\mathrm{L}} \Delta$ is logically valid if and only if, for all interpretation, if at least one conclusion is only false, the premises are also false.

$\leftarrow$*fnt*: The argument $\Gamma \vDash_{\mathrm{L}} \Delta$ is logically valid if and only if, for all interpretation, if at least one conclusion is not only true, the premises are only false.

$\leftarrow$*ntf*: The argument $\Gamma \vDash_{\mathrm{L}} \Delta$ is logically valid if and only if, for all interpretation, if at least one conclusion is only false, the premises are not only true.

$\leftarrow$*ntnt*: The argument $\Gamma \vDash_{\mathrm{L}} \Delta$ is logically valid if and only if, for all interpretation, if at least one conclusion is not true, the premises are not true either.

Theorem. I$\not\lessgtr$ and E$\not\lessgtr$ evaluated with $\leftarrow$*ntf* are valid, but Transitivity is not.

Proof. Suppose that $A \not\lessgtr B$ is not true, that is, $1 \notin i(A \not\lessgtr B)$. This implies, in the semantics we are working on, that it is false, i.e., that $0 \in i(A \not\lessgtr B)$. But if $0 \in i(A \not\lessgtr B)$ and $1 \notin i(A \not\lessgtr B)$ then, by the evaluation conditions of $A \not\lessgtr B$, both $0 \in i(A)$ and $1 \notin i(A)$, and $0 \in i(B)$ and $1 \notin i(B)$. Thus, for any interpretation i, if $1 \notin i(A \not\lessgtr B)$, $0 \in i(A)$ and $0 \in i(B)$. But this means that $A \vdash_{\mathrm{T}} A \not\lessgtr B$ and $B \vdash_{\mathrm{T}} A \not\lessgtr B$ are $\leftarrow$*ntf* -valid.

Suppose now that A is not true, that is, $1 \notin i(A)$. This implies, in the semantics we are working on, that it is false, i.e., that $0 \in i(A)$. Then, by the evaluation conditions of $A \not\lessgtr B$, $0 \in i(A \not\lessgtr B)$. Therefore, since i was arbitrary, $A \not\lessgtr B \vdash_{\mathrm{T}} A$ is $\leftarrow$*ntf* -valid. The reasoning for $A \not\lessgtr B \vdash_{\mathrm{T}} B$ is the same, *mutatis mutandis*.

Finally, there is an interpretation that shows that $A \vdash_{\mathrm{T}} B$ is not $\leftarrow$*ntf*-valid, namely, $i(A) = \{1\}$ and $i(B) = \{0\}$. Thus, while $A \vdash_{\mathrm{T}} A \not\lessgtr B$ and $A \not\lessgtr B \vdash_{\mathrm{T}} B$ are $\leftarrow$*ntf*-valid, $A \vdash_{\mathrm{T}} B$ is not $\leftarrow$*ntf*-valid. □

---

[7] For example, in the presentation of the four definitions of mixed logical consequence proposed by Cobreros et al (2012a), more than two truth values are employed. This raises the question, as one of the reviewers suggests, whether three or four distinct values are being used.

[8] As a reviewer correctly observed, in classical contexts where Transitivity is a valid meta-argument, the relations *tt*, *ts*, *st*, and *ss* are coextensional with $\leftarrow$*ff*, $\leftarrow$*fnt*, $\leftarrow$*ntf*, and $\leftarrow$*ntnt* respectively. However, these distinctions are meaningful only in metatheories where Transitivity need not hold. Under a classical metatheory some definitions are coextensive.

Considering the language of **FDE** and the consequence relation ←*ntf*, we obtain the same valid arguments as those in **ST** logic (for more information, see Barrio, Rosenblatt, and Tajer (2015)).[9] In the resulting logic, Transitivity does not hold. However, all logical truths of classical logic[10] and classically valid arguments of the form $\Gamma \vdash_{\mathbf{L}} A$ (though not all meta-arguments, i.e., arguments of the form "if $\Gamma \vdash_{\mathbf{L}} A$ then $\Delta \vdash_{\mathbf{L}} B$"; Transitivity is an example of this) do hold.[11] One advantage of adopting ←*ntf* is that it is not disjunctive, so with or without t*onk*, it does not seem *ad hoc*. Supposing the validity of certain arguments, such as Transitivity, disjunctive and non-disjunctive logical consequence relations may be coextensive, as we will see later. However, one of the lessons we learn from tackling this problem is that presentation matters. In other words, the way we define logical consequence relations is important, so that our results do not appear *ad hoc*. Finally, we also show here that ←*ntf* is useful so that *tonk* does not lead to triviality and can be considered as a meaningful connective.

Before concluding, a crucial point to address is the apparent equivalence between disjunctive and non-disjunctive definitions of logical consequence. A reviewer has suggested there would be no substantial differences between, for example, Cook's definition (using disjunctive clauses) and our non-disjunctive proposal, since both could be equivalent in a classical framework. However, this alleged equivalence depends on some arguments that are invalid in the logical consequence relations we analyze, such as *st* or ←*ntf*. To clarify this, let us first consider the structure of the definitions.

Recall **TL**'s disjunctive definition of logical consequence. A is a logical consequence of Γ in **TL**, $\Gamma \vDash_{\mathbf{TL}} A$, if and only if either: for every interpretation i, if $1 \in i(B)$ for all $B \in \Gamma$, then $1 \in i(A)$; or for every interpretation i, if $0 \in i(A)$, then $0 \in i(B)$ for some $B \in \Gamma$. Let A be 'X is true'; B 'Y is true'; C 'Y is false'; and D 'X is false'. TL's logical consequence relation could be $(A \rightarrow B) \vee (C \rightarrow D)$. Now, consider the following equivalence proof between the two notions:

I.	$(A \rightarrow B) \vee (C \rightarrow D)$	(TL's logical consequence)
II.	$(\sim A \vee B) \vee (\sim C \vee D)$	(I., Extensionality)
III.	$(\sim A \vee D) \vee (B \vee \sim C)$	(II., Commutativity and Associativity)
IV.	$\sim(\sim A \vee \mathrm{D}) \rightarrow (B \vee \sim C)$	(III., Extensionality)
V.	$(A \wedge \sim D) \rightarrow (B \vee \sim C)$	(IV., de Morgan and Double Negation)

Considering the meanings of A, B, C, and D, logical consequence can be read as

st: The argument $\Gamma \vDash_{\mathbf{L}} \Delta$ is logically valid if and only if, for every interpretation, if the premises are only true, the conclusions are not only false.

[9] Whether the fact that they have exactly the same collections of valid and invalid arguments is enough to say that they are the same logic is something we will not discuss here.

[10] A formula A is a *logical truth* (in a logic **L** presented under a semantics S) if and only if, for each interpretation i (of S), $1 \in i(A)$.

[11] As one reviewer pointed out, the resulting logic does not share the same meta-arguments only if we assume local validity. If we assume global validity, it also shares the same meta-arguments of classical logic and **ST**. For more information, see Barrio, Rosenblatt, and Tajer (2015).

The proof of equivalence requires the following assumptions:

• $A \to B \dashv_{\mathbf{L}} \vdash {\sim}A \lor B$	Extensionality
• $A \lor B \vdash_{\mathbf{L}} B \lor C$	Commutativity
• $(A \lor B) \lor C \vdash_{\mathbf{L}} A \lor (B \lor C)$	Associativity
• ${\sim}({\sim}A \lor B) \vdash_{\mathbf{L}} (A \land {\sim}B)$	de Morgan
• ${\sim}{\sim}A \vdash_{\mathbf{L}} A$	Double Negation
• If $A \vdash_{\mathbf{L}} B$ and $B \vdash_{\mathbf{L}} C$ then $A \vdash_{\mathbf{L}} C$	Transitivity

While these definitions are equivalent in a classical metalanguage, such equivalence presupposes the validity of arguments, like Transitivity, that are invalid in the logical consequence relations we have outlined. Thus, the equivalence between disjunctive and non-disjunctive forms, which critically depends on this property. Finally, although the proof of the previous theorem in the metalanguage uses Transitivity, this does not guarantee that the defined relations (such as $_{\leftarrow}ntf$) inherit the same property. In fact, the non-transitivity of $_{\leftarrow}ntf$ and *st* clearly exemplifies how we typically use a slightly stronger logic in the metatheory. In summary, the equivalence between disjunctive and non-disjunctive definitions appears to depend on Transitivity, at least in the standard proofs or frameworks considered here. However, it remains an open question whether such equivalence could be established independently of Transitivity.

4. *Bonus*

Mixed connectives such as *tonk* raise the question of whether they are of the same type as some of the connectives that were used to define them or whether they are of a new type. In the case of *tonk*, the question is whether it is a disjunction, a conjunction, both or neither, or perhaps it is a completely new type of connective. However, how do you know whether a connective is, say, a disjunction?

Before answering this question, we need to define some concepts. Following Estrada-González and Nicolás-Francisco (2024), in a Dunn semantics, an expression of the form $v_i \in (A)$ with $v_i \in \{1, 0\}$ is called a *Dunn atom*. Let $v_i \in (A)$ be a Dunn atom: we will say that $v_j \notin i(A)$, with $v_i, v_j \in \{1, 0\}$ and $v_i \neq v_j$, is its *Boolean counterpart*. For example, the following cases (considered horizontally) are Boolean counterparts of each other:

$$0 \in i({\sim}A) \qquad\qquad 1 \notin i({\sim}A)$$

$$1 \in i(A \lor B) \qquad\qquad 0 \notin i(A \lor B)$$

A *tweaking* is a modification in the evaluation conditions of a connective in which the only changes consist of substituting Dunn atoms for their Boolean counterparts.

Let us now consider the logic **FDE**. **FDE** can be presented by means of a language L, constructed in the usual way, with the following connectives: ~, ∧, ∨. The evaluation conditions and tables are the same as those presented in Section 2. The logical consequence relation of

FDE is as follows: Let A and Γ be a formula and a set of formulas of L, respectively. A is a logical consequence of Γ in **FDE**, $\Gamma \vDash_{\mathbf{FDE}} A$, if and only if, for every interpretation i, if $1 \in i(B)$, for all $B \in \Gamma$, $1 \in i(A)$.

Estrada-González and Nicolás-Francisco (2024) say that a connective $\odot$ is a *classically clear case* of negation/conjunction/disjunction/conditional, if:

1. The evaluation conditions of $\odot$ are the negation/conjunction/disjunction/conditional conditions of **FDE**; or
2. The evaluation conditions of $\odot$ are obtained from a *tweaking* of the evaluation conditions of **FDE**.

A connective $\circledast$ is a *negation/conjunction/disjunction/conditional* if and only if there is a clear case $\odot$ of negation/conjunction/disjunction/disjunction/conditional such that:

$$\odot(A_1,\ldots,A_n) \dashv_{\mathbf{L}}\vdash \circledast(A_1,\ldots,A_n)$$

We believe that few would doubt that the connectives $\vee$ and $\wedge$ are classically clear cases of disjunction and conjunction, respectively. Although the tables with four interpretations look different from the classical ones, the evaluation conditions of $\vee$ and $\wedge$ are the same. We can state that the disjunction $\vee$ and the conjunction $\wedge$ satisfy 1. However, the evaluation conditions of $\not\lessgtr$ do not correspond to those of any **FDE** connective and cannot be obtained by tweaking its evaluation conditions. Then $\vee$ and $\wedge$ are classically clear connectives, unlike $\not\lessgtr$.

Now let us consider a version of the **FDE** disjunction with the three admissible interpretations {1}, {1,0},{0}[12], i.e.,

$A \vee B$	{1}	{1, 0}	{0}
{1}	{1}	{1}	{1}
{1,0}	{1}	{1, 0}	{1, 0}
{0}	{1}	{1, 0}	{0}

It is easy to see that the following arguments are $\leftarrow ntf$ valid

$$A \not\lessgtr B \vdash_{\leftarrow ntf} A \vee B \quad \text{and} \quad A \vee B \vdash_{\leftarrow ntf} A \not\lessgtr B$$

That is, it holds that

$$A \not\lessgtr B \dashv_{\leftarrow ntf}\vdash A \vee B$$

[12] This table corresponds to the disjunction of logics such as **LP**, when presented with Dunn semantics. Thus, it is easy to see that both classical logic, **FDE** and **LP** have the same evaluation conditions for their connectives, but differ in their tabular presentation.

We can say that, in this approach, ≸ is a disjunction. Since the truth conditions are the same for ∨ and for ≸, the proof is trivial. Both $A≸B$ and $A \vee B$ have only one interpretation where they are not true, $0 \in i(A)$ and $0 \in i(B)$. Hence, when $1 \notin i(A \vee B)$ then $0 \in i(A≸B)$, and vice versa.

Now let us consider a version of the conjunction of **FDE** with the three admissible interpretations {1}, {1,0}, {0}[13] , i.e.,

$A \wedge B$	{1}	{1, 0}	{0}
{1}	{1}	{1, 0}	{0}
{1,0}	{1, 0}	{1, 0}	{0}
{0}	{0}	{0}	{0}

It is easy to check that the following arguments are ←*ntf*-valid

$$A≸B \vdash_{\leftarrow ntf} A \wedge B \quad \text{and} \quad A \wedge B \vdash_{\leftarrow ntf} A≸B$$

That is, it holds that

$$A≸B \dashv_{\leftarrow ntf}\vdash A \wedge B$$

We can state that, in this approach, ≸ is also a conjunction. Since the falsity conditions are the same for ∧ and for ≸, it is enough to look at the truth tables to see that $1 \notin i(A \wedge B)$ if and only if $1 \notin i(A)$ or $1 \notin i(B)$. Finally, in these same interpretations, $0 \in i(A≸B)$. Then, $A≸B \vdash_{\leftarrow ntf} A \wedge B$. On the other hand, $1 \notin \mathrm{i}(A≸B)$ only in the case where $0 \in i(A)$ and $0 \in i(B)$. In the same interpretation, $0 \in i(A \wedge B)$. Then $A \wedge B \vdash_{\leftarrow ntf} A≸B$.

It is not strange that ≸ is a conjunction and a disjunction at the same time, given its introduction and elimination rules. However, in Estrada-González and Nicolás-Francisco's method, logical consequence plays a decisive role in the clarification of this type of mixed connectives. For example, if we were to evaluate $A≸B \dashv_{\leftarrow ntf}\vdash A \vee B$, and $A≸B \dashv_{\leftarrow ntf}\vdash A \wedge B$ with a truth-preserving logical consequence relation, the result would be that ≸ is only a disjunction, but not a conjunction. However, to show our result, we use homogeneously the same consequence relation of the theory, i.e. ←*ntf*-validity. Therefore, ≸ here is both a disjunction and a conjunction. The argument that *tonk* is a disjunction and a conjunction is not conclusive, certainly. This is because the conclusion depends on ←*ntf*-validity, and although we have argued that this is a good notion to work with in this context, different logical consequence relations may give different results about what kind of connective *tonk* is.

[13] This table corresponds to the conjunction of logics such as **LP**, when presented with Dunn semantics.

5. *Conclusion*

In recent times, many people emphasize the limits of logic and formalization, appealing to the most diverse limitative theorems. With this paper we wanted to give an example that, in logic, with enough care and freeing oneself from some unnecessary assumptions, one can have the cake and eat it.

When many have considered a formal result to be definitive, they do so on the basis of various assumptions, which are often implicit. In the case of *tonk*, for example, the validity of Transitivity in the consequence relation was an implicit assumption that distorted *tonk*'s picture. Probably the lesson to be drawn here is that there are no strange or bad connectives per se, only connectives that may be incompatible with certain languages and certain consequence relations. This lesson is a generalization of Belnap's work which, as we said, suggests that *tonk* is a trivializing connective only in transitive logics.[14]

We showed that, without artificial motivation, it is possible to obtain a logical consequence relation with which *tonk* does not trivialize. Also, we proved that it is possible to define *tonk* with fewer admissible interpretations than those used by Cook. Finally, as a bonus, we presented a way of identifying *tonk* as a connective that is both a disjunction and a conjunction.

Acknowledgements

We would like to thank the members of Lógica M∃*x*∀ for their comments on previous versions of the typescript. The second author thanks especially Manuel Eduardo Tapia-Navarro and Dubian Cañas-Mora for their suggestions to improve some passages. We also thank the reviewers for their comments, which significantly improved the final version of the text. This work was supported by the projects CBF2023-2024-55 and PAPIIT IN406225.

References

Barceló, A. A., (2008). Patrones inferenciales. *Crítica: Revista Hispanoamericana de Filosofía*, *40*(120), 3–35. https://doi.org/10.22201/iifs.18704905e.2008.993

Barrio, E., Rosenblatt, L., & Tajer, D. (2015). The logics of strict-tolerant logic. *Journal of Philosophical Logic*, *44*(5), 551–571. https://doi.org/10.1007/s10992-014-9342-6

Belnap, N. D. (1962). Tonk, Plonk and Plink. *Analysis*, *22*(6), 130–134. https://doi.org/10.2307/3326862

Béziau, J.-Y. (2001). Sequents and bivaluations. *Logique et Analyse*, *44* (176), 373–394.

Buacar, N. M. (2018). Tonk y la concepción semanticista del significado. *Cuadernos de filosofía*, *70*, 5–18. https://doi.org/10.34096/cf.n70.6498

[14] In the same spirit, Ramírez-Cámara and Estrada-González (2021) also argue that a 'nasty' connective such as *knot* (which, unlike *tonk*, invalidates many classically valid argument schemes even though it looks like a mere generalized identity connective) is not so nasty in non-reflexive logics, either.

Carnap, R. (1943). *Formalization of logic*. Harvard University Press.

Chemla, E., Égré, P., & Spector, B. (2017). Characterizing logical consequence in many-valued logic. *Journal of Logic and Computation*, *27*(7), 2193–2226. https://doi.org/10.1093/logcom/exx001

Church, A. (1944). Review of *Formalization of Logic*, by R. Carnap. *The Philosophical Review*, *53*(5), 493–498. https://doi.org/10.2307/2181359

Church, A. (1953). Non-normal truth-tables for the propositional calculus. *Boletín de la Sociedad Matemática Mexicana*, *10*(1-2), 41–52.

Cobreros, P., Égré, P., Ripley, E., & van Rooij, R. (2012a). Tolerant, classical, strict. *Journal of Philosophical Logic*, *41*, 347–385. https://doi.org/10.1007/s10992-010-9165-z

Cobreros, P., Égré, P., Ripley, E., & van Rooij, R. (2012b). Tolerance and Mixed Consequence in the S'valuationist setting. *Studia logica*, *100*, 855-877. https://doi.org/10.1007/s11225-012-9422-y

Cook, R. T. (2005). What's wrong with Tonk (?). *Journal of Philosophical Logic*, *34*(2), 217–226. https://doi.org/10.1007/s10992-004-7805-x

da Ré, B., Szmuc, D., Chemla, E., & Égré, P. (2023) On Three-Valued presentations of classical logic. *Review of Symbolic Logic*, *16*(3) 1–23.https://doi.org/10.1017/S1755020323000114

Dunn, J. M. (1976). Intuitive semantics for First-Degree Entailment and "coupled trees". *Philosophical Studies, 29*, 149–168. https://doi.org/10.1007/BF00373152

Dunn, J. M. (2000). Partiality and its dual. *Studia Logica*, *66*, 5–40. https://doi.org/10.1023/A:1026740726955

Estrada-González, L. (2019). The Bochum Plan and the foundations of contra-classical logics, *CLE e-Prints*, *19*, 1–21.

Estrada-González, L. (2022). Dunn semantics for contra-classical logics, *EPTCS*, *358*, 298–309.

Estrada-González, L., & Nicolás-Francisco, R. A. (2024). Connexive negation. *Studia Logica*, *112*, 511–539. https://doi.org/10.1007/s11225-023-10078-x

Omori, H., & Sano, K. (2015) Generalizing functional completeness in Belnap-Dunn logic, *Studia Logica, 103*(5), 883–917. https://doi.org/10.1007/s11225-014-9597-5

Prior, A. N. (1960). The runabout inference-ticket. *Analysis*, *21*(2), 38–39. https://doi.org/10.2307/3326699

Ramírez-Cámara, E., & Estrada-González, L. (2021). Knot is not that nasty (but it is hardier than tonk). *Synthese*, *198* (22),5533–5554. https://doiorg.pbidi.unam.mx:2443/10.1007/s11229-019-02498-x

Ripley, E. (2015). Anything goes. *Topoi*, *34* (1), 25-36. https://doi.org/10.1007/s11245-014-9261-8

Shoesmith, D., & Smiley, T. (1978). *Multiple Conclusion Logic*. Cambridge University Press.

Stevenson, J. T. (1961). Roundabout the runabout inference-ticket. *Analysis*, *21*(6), 124–128. https://doi.org/10.1093/analys/21.6.124

Teijeiro, P. (2020). Not a Knot. *Thought: A Journal of Philosophy*, *9*, 14–24. https://doi.org/10.1002/tht3.437

Wansing, H., & Shramko, Y. (2008). Suszko's thesis, inferential many-valuedness, and the notion of a logical system. *Studia Logica*, *88*(3), 405–429. https://doi.org/10.1007/s11225-008-9111-z

Wintein, S. (2016). On all Strong Kleene generalizations of classical logic. *Studia Logica*, *104*, 503–545. https://doi.org/10.1007/s11225-015-9649-5

LUIS ESTRADA-GONZÁLEZ is a Mexican philosopher. He is a Permanent Researcher at the Institute for Philosophical Research of the National Autonomous University of Mexico (UNAM), and, since November 2024, Director of the Institute. Member of the Mexican National System of Researchers (SNII, level III), the Mexican Academy of Sciences, and the Logicians' Liberation League. Associate Editor of *Studia Logica* and member of the International Editorial Board of *Stoa*.

ADDRESS: Institute for Philosophical Research, National Autonomous University of Mexico, Circuito Mario de la Cueva s/n, Ciudad Universitaria, Mexico City, 04510, Mexico. – Email: loisayaxsegrob@comunidad.unam.mx – ORCID: https://orcid.org/0000-0002-1466-0240

CHRISTIAN ROMERO-RODRÍGUEZ is a Colombian philosopher. Currently postdoctoral researcher in a SECIHTI project on logical pluralism, based at the Institute for Philosophical Research of the National Autonomous University of Mexico (UNAM). His research focuses on non-classical logics, especially paraconsistent and empty logics.

ADDRESS: Institute for Philosophical Research, National Autonomous University of Mexico, Circuito Mario de la Cueva s/n, Ciudad Universitaria, Ciudad de México, 04510, México. – Email: Christian.romero@outlook.com – ORCID: https://orcid.org/0000-0002-7211-3392