

Signs, figures and time: Cavaillès on “intuition” in mathematics¹

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ABSTRACT: This paper is concerned with Cavaillès’ account of “intuition” in mathematics. Cavaillès starts from Kant’s theory of constructions in intuition and then relies on various remarks by Hilbert to apply it to modern mathematics. In this context, “intuition” includes the drawing of geometrical figures, the use of algebraic or logical signs and the generation of numbers as, for example, described by Brouwer. Cavaillès argues that mathematical practice can indeed be described as “constructions in intuition” but that these constructions are not imbedded in the space and in the time of our Sensibility, as Kant believed: They take place in other structures which are engendered in the history of mathematics. This leads Cavaillès to a critical discussion of both Hilbert’s and Brouwer’s foundational programs.

Key words: sign, symbol, figure, time, intuition, Cavaillès, Hilbert, Brouwer, Kant.

This paper is concerned with Cavaillès’ account of “intuition” in mathematics. The term “intuition”, as used by Cavaillès in this context, is misleading. In particular, what Cavaillès calls intuition includes the mental act of counting, the use of signs (algebraic or logical signs) and the drawing of geometrical figures. In geometry, that places on the same level the drawing of a figure and the proof of a theorem in so far as it uses signs, or symbols (I will take the two words as equivalent). The expression, the writing of a proof is as much part of intuition as the geometrical figure. This apparent polysemy of the term “intuition” comes from the fact that Cavaillès first relies on Kant’s theory of “construction in intuition”, which brings together arithmetical, geometrical and algebraic constructions. Cavaillès then uses Hilbert’s analysis to apply Kant’s theory to modern mathematics.

My aim, first, is to explain Cavaillès’ position, which, I believe, is itself of interest. I will not, however, follow step by step the various texts in Cavaillès’ work that are concerned with this question of intuition.² I will rather concentrate on a few points and try to reconstruct precisely how Cavaillès arrives at his conclusions. I will not attempt to defend Cavaillès’ thesis. I simply want to make it clear and, perhaps, useful for a

¹ This paper has been revised on several points after the useful report of an anonymous referee.

² I have done this elsewhere (Cassou-Noguès, 2001). I will mainly refer to five texts: on Kant, Cavaillès (1938), in Cavaillès (1994), p. 34-40; on Hilbert, *idem*, p. 99-103; on Cavaillès’ own conception of intuition, *idem*, p. 179-191; on Kant, “Transfinito et continuo”, in Cavaillès (1994), p. 469-472; on mathematical symbols, Cavaillès (1947), in Cavaillès (1994), p. 520-521.



contemporary philosopher of mathematics. Anyone who has read some of Cavailles' texts will recognize that this is in itself a demanding piece of work.

With this case study, I also want to illustrate the general orientation of Cavailles' writings and of a good part of epistemology "à la française". By this unfortunate expression, I understand a philosophical tradition going from Brunschvicg to Bachelard, Cavailles, Gonthier, Lautman, more recently, to Desanti and, to some extent, Althusser and Foucault.³ Brunschvicg once said that history (and he meant history of science, mathematics and physics) is the philosopher's laboratory. History of science then becomes a field where one can test, discuss and rectify the concepts and the problems of traditional philosophy. Brunschvicg's sally sums up the perspective of the authors mentioned above. They carefully study various periods of the history of sciences while never taking their eye off of other matters. What they seek to understand, by studying the history of mathematics, or physics, is "Reason", or "Rationality", how the human mind works, what knowledge is and what relationship the mind has to its objects and to the external world. The analysis of mathematical theories is just a step towards more general questions. In that sense, history of science is not so much the object of study as a medium for philosophy, its laboratory precisely.

Born in 1903, Jean Cavailles died at the beginning of 1944. His work bears on the history and philosophy of mathematics from the early nineteenth century to 1938. The last result Cavailles discusses is Gödel's proof of relative consistency for the continuum hypothesis and the axiom of choice. His books include *Remarques sur la formation de la théorie abstraite des ensembles* (1938), *Méthode axiomatique et formalisme* (1938) and *Sur la logique et la théorie de la science*, a manuscript written in 1943 and published posthumously in 1947, investigating the philosophy of logic, from Kant to Carnap and Husserl. At the time of their publication, his books gained a wide audience, as shown by reviews signed by E. Beth, H. Cartan, A. Church, A. Fraenkel. However, in part because of the difficulty of his language, in part because of the widening gap between analytical and continental philosophy, his work, outside France, gradually fell into oblivion. Contrary to those of Bachelard, his books have not been translated into English.⁴

Cavaillès' concern with intuition in mathematics starts with his first books (in 1938) but it is still there in later papers such as 'Transfinité et continu' (from 1941). Cavailles' account stems from a comparison between Kant and Hilbert. I will follow this path. I start by presenting Kant's position (in 1.) to make clear the background of Cavailles' analysis. I then recall some of Hilbert's remarks on mathematical practice (in 2.), which Cavailles uses to apply Kant's theory of intuition to modern mathematics. I discuss three points in particular: in 3. Cavailles' interpretation of the status of logic in contemporary mathematics; in 4. and 5., the notion of "Combinatorial space" and the critical examination by Cavailles of Hilbert's account of the role of symbols in

³ On the relation of Althusser and Desanti to the epistemology "à la française", see Cassou-Noguès, forthcoming. On Foucault and Cavailles, see D. Hyder (2003).

⁴ There is very little secondary literature in English. In French, two books really concentrate on Cavailles' philosophy: Sinaceur (1994) and Cassou-Noguès (2001).

mathematics; in 7. the relation of intuition to time, with Cavaillès’ argument against Brouwer.

1. The *Critique of pure reason* begins with the problem of synthetic *a priori* judgements, of which mathematical propositions are the paradigm (Kant 1781-1787, introduction, IV). Kant, in the context of Aristotelian logic, considers all propositions to be of the form subject-predicate. A judgement is said to be analytic when the predicate is contained in the concept of the subject: « a tall man is tall ». A judgement is said to be synthetic when the predicate is not contained in the concept of the subject. This is the case of mathematical judgements: « $5 + 7 = 12$ », « the sum of the angles of a triangle is equal to a flat angle ». A triangle is a closed figure, with three angles and three sides. If we analyse the concept of a triangle, the concept of the angles of a triangle, can we deduce, purely from these concepts, that the sum of the angles is π ? Kant holds we cannot. The judgement is therefore synthetic. Mathematical judgements are synthetic though they are independent of empirical experience and, in this sense, *a priori*. The task is to explain how such judgements, synthetic and *a priori*, are possible.

Since the predicate is not contained in the concept of the subject, there must exist some external basis for the predication. This basis cannot be logic. Logic will only produce analytic judgements. « Men are mortal; Socrates is a man; therefore Socrates is mortal ». The predicate, mortal, is contained in the concept of the subject, Socrates, if, indeed, Socrates is a man. According to Kant, logic (that is Aristotelian logic) is only a means to produce analytic judgements. Synthetic judgements do not rest on this basis. Synthetic judgements rest on a construction in intuition. We draw a triangle, we draw a straight line passing through one summit and parallel to the opposite side, then we show that the sum of the three angles coincides with a flat angle. In the Euclidean tradition, all geometrical propositions are based on such drawings. The concepts they involve must be illustrated by figures or « constructed » in space, and it is on the basis of this construction that one can link the predicate to the concept. We cannot understand, merely by analysing the concepts involved, why the sum of the angles of a triangle is equal to a flat angle. We can only see it by means of a drawing. Obviously, a difficulty here is that the figure we draw is singular, whereas the proposition we state is universal and should apply to all figures. Kant is aware of this point (Kant 1781-1787, Discipline of pure reason, I).

Arithmetic also relies on constructions in intuition but those are independent of space. To calculate the sum « $5 + 7$ », we place five things or, say, five apples, beside seven other apples, and then we count and find there are twelve apples. The particular nature of the things that we count does not matter in this process. Indeed, we do not even need to consider things existing in space. We could count concepts. The process only requires that the objects be distinguishable from each other and counted, one by one, in a succession. What does matter is that the act of counting takes place in time. Arithmetic depends only on our inner temporality, on the temporality of our mental life. In this sense, arithmetic is based on a construction in time.

Eventually, mathematics, as far as it rests on constructions in intuition, calls upon three faculties of the mind. First, our Sensibility gives us a multiplicity in space and in time. Second, we must form a representation of this multiplicity. We must go through the elements and consider the construction as a whole. This is what Kant call the synthesis of intuition. It is the work of Imagination. Third, we apply concepts to these representations and establish judgments. At this point, we recognize the figures as a triangle, a straight line, and we state the equality of the sum of the angles with a flat angle. Here, we use our Understanding.

According to Kant, mathematical propositions suppose a kind of experiment, a « construction », in space and in time. Space is the form of our intuition of the external world. All external objects are apprehended in space. Time is the form of our intuition of our mental life. All internal events are organized inside time. Thus, space and time precede experience. They constitute the setting, internal and external, in which concrete experience takes place. This is why mathematical propositions are valid *a priori*, independently of concrete experience.

Kant also mentions an “algebraic construction”, which is carried out in space, just as geometrical constructions, but uses symbols instead of figures (*idem*). However, this passage, in the *Critique of pure reason*, is rather allusive. Cavallès will develop this idea using Hilbert’s insights on mathematical practice.

2. Cavallès will rely on various remarks from Hilbert’s papers to rectify the Kantian conception and apply it to modern mathematics. The underlying idea is that the chains of formulas by which formalization represents mathematical proofs, really are constructions in intuition and, therefore, can be compared with geometrical figures in Kant’s theory. That may seem far-fetched. However, Hilbert does make explicit references to Kant:

“[...] we find ourselves in agreement with the philosophers, especially with Kant. Kant already taught —and indeed it is part and parcel of his doctrine— that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone; that is why the efforts of Frege and Dedekind were bound to fail. Rather as a condition for the use of logical inferences and the performance of logical operations, something must be given to our faculty of representation, certain extra logical concrete objects that are intuitively present as immediate experience prior to all thought.” (Hilbert 1925, p. 376)⁵

The lines that follow this paragraph from “On the Infinite” make it clear that the “concrete objects”, to which Hilbert alludes, are mathematical signs. In fact, Hilbert seems to refer indiscriminately to the objects, such as the “dash”, of finitary mathematics, and to the symbols used in transfinite mathematics, which also make the objects of metamathematics.

The question is to understand the reference to Kant in this context. Kant introduces the notion of construction in intuition in order to explain why mathematics goes beyond logic, which, at this time, is Aristotelian logic. Hilbert argues that one cannot give a proper foundation to mathematics by reducing it to logic, as Frege, Rus-

⁵ See also Hilbert 1927, p. 464-465; 1931, p. 1150.

sell or, in a way, Dedekind tried to do. Mathematics is irreducible to logic and that, says Hilbert, is because, as Kant has seen, mathematical inferences depend on an intuition: concrete objects are being given to us, on which mathematical inferences rely. It is true that “logic” is no longer Aristotelian logic. The analogy, which Hilbert hopes to establish between his problem and Kant’s, could be far fetched. Nevertheless, Hilbert seems to consider that signs, seen as “concrete objects”, have the same role as the geometrical “figures” of Kant’s constructions. Let us, following Cavaillès, take Hilbert up to his words and try to see in what sense the symbolic manipulations, which represent mathematical proofs in formal systems, can be considered as constructions in intuition.

First, there is an ambiguity in the relation of formalization to mathematical practice. It sometimes seems as though formal systems were artificial objects, created for the purpose of foundation, and without relation to the mathematician’s practice. Hilbert himself speaks of “replacing” contentual inferences by manipulations of signs, “converting” the propositions that constitute mathematics into formulas, so as to obtain “in place of” the contentual mathematics an inventory of formulas (Hilbert 1925, p. 381; 1927, p. 465; 1925, p. 381). Here, this inventory of formulas seems to remain foreign to genuine mathematics, and there is no reason to compare mathematical practice with a manipulation of signs. However, the case is more complicated.

The task of formalization is to make explicit the rules and the axioms that govern mathematical practice but are left implicit in mathematical practice. By doing so, formalization operates a rectification of mathematical practice. It makes visible the different steps of a proof, which would be overlooked in practice, and their conformity to the rules of deduction. Indeed,

“this formula game enables us to express the entire thought content of the science of mathematics [...], in such a way that, at the same time, the interconnections between the individual propositions and facts become clear.” (Hilbert 1927, p. 475)

The correct expression of mathematical thoughts is to be found in the “proof figures” of formal systems. These “figures” reflect the genuine thoughts that constitute mathematics. Formulas are the “images” [*Abbilder*] of thoughts (Hilbert 1923, p. 1138; 1927, p. 465). Thus, to formalize a theory is to make explicit the different thoughts that constitute a proof, their relationships and the rules they obey. Formalization can be considered as an inquiry into the functioning of the mathematical mind:

“The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. Thinking, it so happens, parallels speaking and writing: we form statements and place them one behind the other.” (Hilbert 1927, p. 475)⁶

I will come back to this “parallelism” between thoughts and formulas. Now the point is that, if mathematical practice is merely rectified by formalization, it must already include a formula game (to use Hilbert’s words), a manipulation of signs, which

⁶ See also Hilbert 1931, p. 1156, and Hilbert 1922, p. 1120: “To proceed axiomatically means in this sense nothing else than to think with consciousness”.

formalisation systematises. Indeed, as early as 1900, Hilbert insists on the role of signs in mathematical work. In his famous lecture on mathematical problems, he stresses the importance of signs in various branches of mathematics, algebra, analysis, and geometry. For example, given an equation to solve, the mathematician will “experiment” on the formulas, transform the initial formula and find new expressions that will indicate a solution. It is through this game with formulas that the mathematician can go forward. The formula game, in arithmetic or in algebra, has the same function as the drawing of figures in geometry: it makes progress possible.

“[...] we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behaviour of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry” (Hilbert 1900, p. 1101)

There is then an analogy between arithmetical formulas and geometrical figures:

« *Die arithmetischen Zeichen sind geschriebene Figuren, und die geometrischen Figuren sind gezeichnete Formeln* »

“The arithmetical signs are written figures, and the geometrical figures are designed formulas”⁷

We have two texts from Hilbert, “On the infinite” (1926) and “Mathematical Problems” (1900), widely separated in time but going in the same direction. The first text refers to Kant’s notion of construction in intuition. The second compares the manipulations of signs, in various branches of mathematics, to the drawing of figures in geometry. They may seem to entitle Cavailles to consider the formula game, this manipulation of signs that underlies mathematical practice, as a construction in intuition in Kant’s sense. However this comparison leads to several problems.

3. In his lecture of 1900, Hilbert argues that arithmetic, or algebra, depends on a work on formulas just as geometry depends on the drawing of figures. If we are looking for the proof of a theorem or investigating some abstract structures, we will write short notes, try to imagine new expressions, to modify the formulas we already have. Although they might be at the margins of our attention, we will be working with, and on, signs. However, it seems there is a difference between the role of signs, in mathematics generally, and the role of figures in geometry. A sign denotes an object, « 5 » denotes the number 5, and it is this object, or its domain, that the mathematician investigates. On the other hand, in geometry, the figure seems to be, or is considered by Kant as, the object of geometry. It is because we see a triangle on our drawing and make the sum of its angles coincide with a flat angle that we can prove our theorem. The construction in intuition, as described by Kant, concerns the objects of geometry, whereas the formula game, as it is discussed by Hilbert, concerns the expressions of mathematics, not its objects. How can the two be compared? This point will require a lengthy explanation. So I will start with another difficulty: the status of logic.

Kant, referring to Aristotelian logic, distinguishes between the analytic judgements of logic and the synthetic judgements of mathematics. Mathematical judgments rest on a construction in intuition. On the other hand, logical judgements rely solely on

⁷ Hilbert (1900), German text p. 295 and English translation modified p. 1100.

our faculty of thinking. Thus, conversely, logic becomes a means to isolate a faculty of thinking, an Understanding, independent of intuition. Moreover, Kant takes the different forms of syllogism to denote the different categories, that is the functions, of our Understanding. Now, as Kant, Hilbert refuses to reduce mathematics to logic. But, contrary to Kant, he does not separate mathematics and logic. Mathematics and logic are bound together in formal systems, which comprise logical and mathematical axioms, logical and mathematical rules of inference. Hilbert makes this point explicit in his 1904 paper, which gives the first outline of the formalist program (Hilbert 1904, p. 100-101). Obviously, Hilbert argues, mathematics presupposes logic, since it needs logical rules for inference, but, as Poincaré has shown, logic also presupposes mathematics, for logical rules cannot be stated without appealing to mathematical notions, such as that of number. Therefore, mathematics and logic are to be redefined and rebuilt together through formalization. Logical inferences will then depend on a manipulation of signs, on a construction in intuition, in the same manner as mathematical inferences.⁸ In a word, logic rests on intuition. Now, if one compares Kant’s position to Hilbert’s or, so to speak, translates Hilbert’s position into the Kantian system, it appears that Hilbert loses the means to isolate an Understanding, a faculty of thinking independent of sensibility. Can we maintain such a faculty of the mind when we have no way to isolate its products? Can we assume that thought is distinct from intuition if we cannot distinguish the work of thought?

That is the first point of Cavailles’ critical reading of Kant and Hilbert.⁹ Since Hilbert makes logic depend on a construction in intuition, he loses what was for Kant an “Ariadne’s thread” towards our Understanding. The position of an Understanding independent of Sensibility, the position of thoughts distinct from formulas cannot be maintained. It is to be noted that Cavailles does not refer to a reflexive experience. Do we “feel” that we can think without using formulas? This is not the question. The question is: can we isolate the pure product of a faculty of thinking? According to Cavailles, Hilbert’s answer to this question is negative. Logic rests on a manipulation of signs and, in that sense, depends on sensibility. It can no longer be seen as the product of our Understanding. And, therefore, we have no reason to maintain the position of an Understanding distinct of Sensibility. As we saw earlier, Hilbert himself alludes to a parallelism between formulas, on the paper, and thoughts, in the mind. He seems therefore to admit that there can be no thought without a corresponding formula, no thought that is not expressed in a formula. But this is not enough. We have no reason to maintain that there are thoughts in the mind since, by themselves, they produce nothing. Since we cannot distinguish the pure work of our thinking, we have no reason to assume that there are pure thoughts, distinct from their expression in

⁸ That is also apparent in the text quoted above from “On the infinite” where Hilbert seems to relate “logical inferences” to a construction in intuition.

⁹ Cavailles (1938a), in Cavailles (1994), p. 100-101.

formulas. As Cavaillès put it in a short paper given at the “Congrès Descartes” in Paris, in 1937, thought is “immanent” to the formula.¹⁰

Of course, it will be a problem for Cavaillès to describe precisely the relation between thoughts and their expression in symbols. In fact, this new problem replaces what was in the Kantian system the problem of the relation between Understanding and Sensibility (a problem which was solved by the theory of “Imagination” and “Schematism”): “The difference with Kant is that there are no purely logical thoughts. Logic is only an aspect of all thinking really fecund, an aspect that cannot be isolated for itself. In this context, the problem of the junction between abstract thoughts and intuition is no longer posed, at least no longer at the same place.”¹¹

In fact, one could argue that it is that problem (to describe the relation between formulas and thoughts, intuition and concepts) that, in his last paper, from 1941, Cavaillès sees as the “fundamental problem of mathematical philosophy.”¹² The difficulty is that, although thinking is not an act in our mind that could be set apart from its expression, one cannot simply identify thoughts and their expression. Thoughts are not formulas, but thoughts are not something else, existing in the mind independently of their expression or that we could separate from their expression. Thinking is not writing but thinking is not an act distinct from writing. Cavaillès leaves thinking and writing, thoughts and formulas, Understanding and Sensibility in this ambiguous relationship: not the same thing but not something separable.¹³ The problem then remains open and Cavaillès does not seem to have a solution. One must remember that his work was stopped by his premature death (at the age of 41). Nevertheless, by opening this problem, Cavaillès anticipates on analyses that one finds later on in Merleau-Ponty or in the early Derrida.

4. Let us come back to the first difficulty mentioned above: the “experiments” on formulas, as seen by Hilbert, concern only symbols whereas the constructions in intuition, according to Kant, concern the objects of geometry. Here, what is constructed in intuition is the objects referred to. So how can Hilbert maintain an analogy between the use of formulas and the construction of figures? Well, Hilbert denies that the figures we draw are the objects of geometry: according to him, they are symbols as any other mathematical symbol.

Signs, in mathematics, in algebra or in logic, are used according to rules. These rules indicate how to write a formula (rules of formation) and how to deduce formulas from one another (rules of transformation). In the propositional calculus, the rules of

¹⁰ “Réflexions sur le fondement des mathématiques” in Cavaillès (1994), p. 579.

¹¹ Cavaillès (1938) in (1994), p. 100: “La différence avec Kant est qu’il n’y a plus de pensée logique pure, la logique n’est qu’un constituant, non isolable de toute pensée fonctionnant véritablement. Dès lors, le problème de la jonction entre pensée abstraite et intuition ne se pose plus, du moins plus au même endroit.”

¹² “Transfinité et continu” in Cavaillès (1994), p. 273. See also Cassou-Noguès (2001), p. 183 and p. 218.

¹³ In that sense: “Symbols are not there for something else that they would simply represent”. See Cavaillès (1938) in Cavaillès (1994), p. 181 and p. 520.

formation tell us that we can start with a letter A , write next to it, on the same line, \rightarrow , then another letter B , and so on. The rules of transformation tell us that if we have written two formulas of the form A and $A \rightarrow B$, we can write B . In most branches of mathematics, say in algebra, the rules are left implicit. Actually, to make explicit the rules which govern the use of signs amounts to formalizing the theory. But, there is no need to make these rules explicit in order to apply them. Indeed, the use of formulas, which underlies mathematical practice, depends on the application of rules. When we work on a formula, develop an algebraic expression, we do not consider all possibilities opened in the plane of the piece of paper we write on. We could then write anything, any sign in any direction. But we know somehow that signs should be written in line ($A \rightarrow B$, rather than $A \rightarrow B$) and in a certain order ($A \rightarrow B$, rather than $AB \rightarrow$). These rules delineate a realm of possibilities, for writing or for transforming a formula, and it is in this realm of possibilities that our “construction” takes place. Our manipulations on signs are determined by rules of use.

On the other hand, the geometrical constructions, as described by Kant, are determined by the structure of space. We draw various figures and investigate their relationships. But these, according to Kant, solely depend on the structure of the space of our intuition. That is precisely the point Hilbert denies. Geometrical figures are governed by rules just as algebraic symbols. And, if we could enunciate these rules, we would turn geometrical constructions into a proper theory, in which we could prove theorems, as in any other theory:

“The geometrical figures are signs. [...] The use of geometrical signs as a means of strict proof presupposes the exact knowledge and complete mastery of the axioms which underlie those figures. [...] Just as in adding two numbers, one must place the digits under each other in the right order, so that only the rules of calculation, i. e., the axioms of arithmetic, determine the correct use of the digits, so the use of geometrical signs is determined by the axioms of geometrical concepts and their combinations.” (Hilbert 1900, p. 1100-1101)

Hilbert does not investigate the practice of geometry. It is not clear, in Hilbert’s text, why geometrical figures require rules of use, as do formulas in algebra. But I think one could argue for this point along the following lines. After all, a geometrical figure is made of marks, which we consider as points, of long strokes, which we consider as straight lines, of curves, which we consider as circles. A child could make many such lines go through two such points. But this drawing would not be a geometrical figure proper, at least in Euclidean geometry. A child could make indefinite drawings with marks, lines and curves. But these drawings only become geometrical figures when they respect certain rules: only one line can go through two different points, and so on. These rules, which govern the use of marks, lines and curves, in geometrical constructions, would correspond to the rules of formation in other branches of mathematics. To make these rules explicit would be to formalize geometrical constructions. One could then prove a theorem by constructing a figure. There would be no need to give another proof, using the axioms of geometry as they are expressed in logical formalisms. A formalized proof is only a figure that respects certain rules and so would be the geometrical construction.

Logical, algebraic formulas and geometrical figures are governed by rules of use. In a logical calculus, when we have written $A \rightarrow$, we must add another formula, on the same line $A \rightarrow B$. A child could write $A \rightarrow \forall$. But the mathematician cannot. The rules of use exclude certain possibilities that a child could follow. Our “experiments” on formulas are determined by the rules of use and, in comparison to what a child could do, restricted. That means that these experiments on formulas are not carried out in the plane of the paper, or in the space of our sensibility, but in a specialized space, which Cavallès calls a “combinatorial space.”¹⁴ What is space? According to Kant, space is the structure of phenomena given in external Sensibility. It represents all the relationships that external phenomena may have. Now, whereas on the paper on which a child draws, any two signs can coexist at any two places, in a logical calculus, symbols can only coexist on parallel lines and in a certain order. The signs, on the mathematician’s paper, cannot hold all the relationships they have on the child’s paper. Therefore, strictly speaking, the space in which the child draws and the space in which the mathematician writes are different. Mathematical signs belong to an artificial structure, with restricted relationships.

In the “experiments” on formulas, what matters is not all the relationships that signs could have, as in the drawing of a child, but the restricted possibilities given by the rules of use. What that means is exactly that the experiments on formulas belong to a certain “combinatorial space”. The same goes for geometrical figures (which are also made of signs governed by rules of use). That is the second point of Cavallès critical reading of Kant. Mathematical constructions, geometrical figures or logical formulas, take place in “combinatorial spaces”, “abstract spaces”, structures that are not given in our Sensibility but are constituted in the history of mathematics (Cavallès 1994, p. 101). More important than the reference to such ‘abstract’ structures is the idea of a historical genesis of the ‘spaces’, the contexts in which mathematicians use their symbols or draw their figures.¹⁵ A problem which Cavallès will try to tackle in the last part of his thesis will be to describe how the combinatorial spaces of the history of mathematics are engendered from one another.

5. However, Cavallès is also led to a critical examination of Hilbert’s remarks. As we saw earlier, in the article “On the Infinite”, Hilbert himself considers mathematical signs as something immediately given: “concrete objects that are intuitively present as immediate experience prior to all thought”. Now Cavallès argues that the space to which the symbols employed by the mathematician belong, is not the space of our sensibility but an abstract structure that is constituted in the history of mathematics. That might be enough to show that mathematical signs are not immediate data. But there is yet another way to get to this point. Cavallès seems to refer to a remark by Hilbert in the lecture of 1900, “The Problems of Mathematics”. Hilbert notes that the mathematical symbols should be chosen so as to remind us of their use.

¹⁴ Cavallès (1938a), in Cavallès (1994), p. 101.

¹⁵ There is also a discussion of Cavallès’ notion of combinatorial space in Granger (1960), p. 51.

“To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians.” (Hilbert 1900, p. 1100)¹⁶

Mathematical symbols (of which geometrical figures are only an example) are related to their use, or their meaning. Their very shape, their design reflects their meaning. For example, why do we count using a vertical dash? (like $///$ and not $---$) Surely, because the vertical dash is more visible. It is distinct from its neighbours. It, so to speak, stops the hand that counts. Mathematical signs, Cavaillès believes, all include such a reference to their use.¹⁷ They are not arbitrary objects. They were chosen for a certain purpose and this purpose shows in their very shape, in the way they are designed. Mathematical signs are historical artefacts. They can be seen as “mixtures of sense and intellect.”¹⁸ They are not purely intellectual objects but they cannot be considered as purely sense-objects. For it is a historical process that has designed them as they now are. To consider that mathematical signs are given “prior to all thought”, as Hilbert does, is to shortcut this historical process. Hilbert, according to Cavaillès, is wrong in his famous “At the beginning, there is the sign”. The sign, in mathematics, is not at the beginning but in the middle, a part and a product, of a historical development: “All descriptions of mathematics as a manipulation of objects in space are confronted to this fundamental character of the mathematical symbol: the numeral, the geometrical figure or simply the dash only intervene as part of an activity which is already mathematics. [...] What [formalism] takes as an absolute beginning is only the surreptitious evocation of former acts and mathematical processes.”¹⁹

This point is important for the formalist program. Indeed, Hilbert thought that transfinite mathematics could be formalized and reduced to a “stock of formulas” arranged according to explicit rules. The task was then to prove, with finitary inferences, that these formal systems are free of contradiction. In other words, to prove that one will never find in the stock of formulas that represent classical mathematics a line such as $0 = 1$. However, if mathematical signs are historical artefacts, this stock of formulas itself presupposes the history of mathematics. Even if it were possible to prove finitarily the consistency of mathematics, it is not clear how adequate this foundation would be. For it would already presuppose that mathematics is an activity that somehow

¹⁶ It is to be noted that again Hilbert considers geometrical figures as mathematical signs, alike any other symbol and even exemplary of what mathematical symbols are.

¹⁷ “Dans la configuration du signe, est inscrit le rappel à ses règles d’emploi”, Cavaillès (1938a) in Cavaillès (1994), p. 102.

¹⁸ Cavaillès (1938a) in Cavaillès (1994), p. 102.

¹⁹ “Toutes les comparaisons des mathématiques avec une manipulation spatiale se heurtent à ce caractère fondamental du symbole mathématique, chiffre, figure, même bâton, de n’être là qu’en tant que partie intégrante ou base d’application d’une activité déjà mathématique [...]. Ce que [le formalisme] prend pour commencement absolu n’est qu’évocation subreptice d’actes et d’enchaînements antérieurs” (Cavaillès (1947), in Cavaillès (1994), p. 521). Also relevant is Cavaillès (1938) in Cavaillès (1994), p. 102: Le signe est “le représentant d’autres opérations concrètes, celles-là simplement supposées, mais dont le résultat importe pour l’usage actuel.”

works, develops historically so as to produce this stock of formulas. The formalist foundation presupposes an underlying activity, a historical process, and it is not clear that, by proving the consistency of formal systems, the formalist program recaptures this historical process or makes its foundation of mathematics independent from this historical process. The problem is not simply to show that mathematical theories are free of contradiction but to explain how a certain activity could develop into a historical process.

6. Up to this point, we have directed our attention to geometrical figures and logical formulas, which seemed to be constructed in space. But, earlier, we saw that Kant refers arithmetical constructions to time. The question now is to understand whether Cavallès' argument on geometrical constructions can apply *mutatis mutandis* to arithmetical constructions. If there are constructions in time as there are constructions in space, what is the "time" of these constructions? Is it constituted as the space structure, the "combinatorial space" of geometrical constructions?

Kant argues that arithmetical judgements are synthetic and must be based on a construction in intuition. We know that « $5 + 7 = 12$ », because we count that five apples, (apples or dots or anything) beside seven apples make twelve apples. The result is independent of the nature of the things counted but rests on the succession of the acts of noticing something and adding one. In that sense, arithmetic is independent of our intuition of space and depends only on our inner temporality.

In an early paper, "Intuitionism and formalism" (Brouwer 1912), Brouwer refers to Kant's theory of arithmetic. According to Brouwer, non Euclidean geometries refute the Kantian thesis that space, Euclidean Space, is the *a priori* form of external intuition. We can prove, in the same way, propositions that refer to objects in the Euclidean space and propositions that refer to objects in a non Euclidean Space. Therefore, if these proofs are based on a construction in intuition, the form of this intuition cannot be the Euclidean space. The first thesis, that space is an *a priori* structure, must be abandoned. However, Brouwer takes up the second thesis, that time is the *a priori* form of internal intuition, i. e. the fixed structure of our mental life. Brouwer also agrees with Kant that arithmetic depends on the intuition of time. In his account of the origin of numbers, Brouwer relates arithmetic to a process taking place in the time of "Consciousness". In his later papers, Brouwer develops a philosophy of his own and drops the reference to Kant. Nevertheless, he still maintains that time is an *a priori* form of mental life and that arithmetic relies on processes, "constructions" inside time.

The new rules that intuitionism sets for arithmetical reasoning then depend on two assumptions: first, that all propositions must be based on actual constructions; second, that these constructions are carried out in the time structure of our mental life. It is clear that this second assumption is an essential ingredient for the intuitionist rejection of classical mathematics (e.g. Cantorian set theory). The admissible constructions, for intuitionism, are determined by this criterion: they must be realizable in time.

When discussing Hilbert’s program, Cavaillès argued that the space in which we make our geometrical or algebraic constructions is not an a priori of our sensibility but a structure constituted through the history of mathematics. Can one make a similar statement about time? In other words, can one maintain that the time in which arithmetical constructions are carried out is not the time of our sensibility but another structure? That could imply that, in principle, intuitionism is wrong in its rejection of classical mathematics.

Cavaillès seems to make such a point in his last paper (from 1941). The point is directed against Kant but it applies to Brouwer as well.

“Rejection of infinite numbers: how could one construct in time what requires an infinity of steps? But that is precisely dissociating these steps from their sole function of *elements* and seeing them as *events*.”

“The time of [mathematical] constructions ... should evade all representation and, therefore, all conditions for the accomplishment of the [act]. But would it still be a time? and could one still talk of accomplishment?”²⁰

To me, the conclusion of this paragraph seems to be that the structure in which mathematical constructions are carried out should not be identified with the time of our sensibility. This identification would be a misinterpretation of mathematical processes. Mathematical processes can be described as a succession of steps. But these steps are simply elements in a structure of some kind. To identify this structure with time as we know it is to make them natural events. And, inevitably, it leads to the rejection of “infinite numbers,”²¹ since these cannot be constructed inside time. Cavaillès seems then to indicate that mathematical processes may have their own “time” structure, if one can still (without ambiguity) consider this structure as a “time”.

It is difficult to see what one can make of Cavaillès’ argument. Cavaillès made the same point against Kant, and Hilbert, on the question of space. Mathematical constructions do not take place in the forms of our sensibility but in structures, which must then be generated by the development of mathematics itself.²² On the question of time, that implies that intuitionism is wrong in its rejection of classical mathematics. Intuitionism is wrong to exclude infinite constructions on the ground that they are not realizable in time.²³ However, Cavaillès cannot mean that there are mathematical

²⁰ “Transfini et continu”, in Cavaillès (1994), p. 470: “D’où l’exclusion du nombre infini: comment construire dans le temps ce qui suppose une infinité d’étapes ? mais c’est justement dissocier ces étapes de leur unique fonction d’*éléments* pour en faire des *événements*” (Cavaillès’ emphasis). Also: “Le temps de la construction [...] devrait échapper à toute représentation et par suite à toute condition dans l’accomplissement de sa synthèse. Mais serait-il encore un temps ? et pourrait-on encore parler d’accomplissement.”

²¹ Of course, in reality, intuitionist mathematics incorporates a segment of the class II of ordinals.

²² See also Cavaillès (1938a), in Cavaillès (1994), p. 35, and Cassou-Noguès (2001), p. 69 and sq.

²³ See also Cavaillès (1938a), in Cavaillès (1994), p. 189: “C’est un reste d’attachement à l’a priori logique qui oblige les intuitionistes à leurs interdiction”. Here, on the problème of time, the intuitionist rejec-

“times” in which we can actually carry out arithmetical constructions with an infinite number of steps, such as the successive generating of the ordinal numbers of the second class. Obviously, we cannot carry out such constructions and we do think in time. It is also to be noted that the “combinatorial space” in which take place the geometrical constructions and the formula game, to use Hilbert’s words, was obtained from the space of Sensibility by a sort of restriction. We said that the symbols, in the hands of the mathematician, could not take all the relations that they would have in the drawing of the child. And, from that, we concluded that mathematical symbols belonged in fact to another structure. However, in the case of time, the structure, in which mathematical constructions must take place, should rather be an extension of the time of our sensibility, if it is to make possible the representation of infinite numbers.

The only way that I find to make sense of Cavallès argument about time is to compare it to a remark of Gödel. As is well known, when discussing the axioms of set theory, Gödel introduces the following criteria: a set exists when the multiplicity of its element can be overviewed by an idealized mind. By this criterion, Gödel intends to justify the usual axioms of set theory plus some additional axioms of the infinite (about large cardinals). The criterion is not understood as restricting drastically the universe of sets. The “idealized mind”, to which it refers, is infinite. However, it is difficult to understand how this idealized mind could overview a multiplicity of elements and put them in order (as would be necessary to obtain an ordinal arithmetic) in a timeless process. But, on the other hand, this process cannot take place in the time that we know, for then it could not overview infinite multiplicities. In this context, one could use Cavallès’ suggestion that, in principle, mathematical processes may use other structures, extending time as we know it.

Cavallès and Gödel do not share the same philosophical background. Cavallès is rather hostile to Platonism. Nevertheless, they seem to agree on the principle that mathematical possibilities should not be judged nor restricted with reference to a human mind allegedly given and analysable by itself. (Intuitionism does just that when it restricts mathematical constructions to those that can be realized in the time structure of a human sensibility). This principle, for Gödel, is a consequence of his Platonism. For Cavallès, it comes from a certain conception of mathematics as a historical process. Mathematics, for Cavallès, is an autonomous process, which develops by itself, and which cannot be referred to the world nor to the mind: “Science [in fact, mathematics] is not to be considered as an intermediary between the human mind and the being in itself, depending as much on one as on the other, but as a *sui generis* object, with an essence of its own and an autonomous development.”²⁴

tion of classical mathematics precisely steams from the fact that they maintain the Kantian thesis that time is the form *a priori* of arithmetical constructions.

²⁴ “Pour la première fois peut-être la science n’est plus considérée comme simple intermédiaire entre l’esprit humain et l’être en soi, dépendant autant de l’un que de l’autre et n’ayant pas de réalité propre, mais comme un objet *sui generis*, original dans son essence, autonome dans son mouvement” (Cavallès (1947) in Cavallès (1994), p. 503.

7. My aim was to follow Cavaillès’ discussion of Kant and Hilbert in order to illustrate some aspect of his work. In conclusion, I put forward three points.

(1) The first point concerns the relation of thought to its expression in symbols. Cavaillès argues that it is impossible to consider thought as an act in the mind, isolated from its expression, or, in any way, to set apart thoughts from their expression. This point is to me exemplary of an orientation of epistemology in France in the first half of the 20th century. Here Cavaillès discusses a philosophical problem (the relation of thought to its expression) using the example of mathematics. His discussion is itself of interest (at least, historically, for it anticipates some of Merleau-Ponty’s later developments) but has no specific import for the philosophy of mathematics. It does not help us understand how mathematics works.

(2) Cavaillès argues that the manipulation of symbols, which Hilbert seems to consider fundamental to mathematical practice, can be considered as a construction in intuition in Kant’s sense. However, this play on symbols does not take place in the space of our sensibility but in specific structures (“combinatorial spaces”) that are engendered in the history of mathematics. From here, Cavaillès draws an argument against Hilbert who, just as Kant, considers mathematical symbols as immediate data (“prior to all thought”) whereas, in reality, they are artefacts of the history of mathematics.

(3) Cavaillès then extends his argument to time considered as a medium for arithmetical constructions. The argument is then directed against Brouwer who requires of mathematical constructions that they be realizable in the time of our sensibility. That, according to Cavaillès, is a misinterpretation of the nature of mathematics.

I have tried to give the textual references that underlie Cavaillès’ conclusions. Cavaillès relies on a close reading of Kant, Hilbert and Brouwer but, certainly, does not argue for his conclusions as a contemporary philosopher of mathematics would do. This lack of argumentation may be common to Cavaillès, Bachelard, Brunschvicg and others from the same period. It seems that what was important, more important than giving arguments, was to show how one’s conclusions stem from mathematics (or, at least, from the reflections of mathematicians) and, in this way, belong to a “mathematical spirit”. Nevertheless, the conclusions are there and will hopefully prove useful.

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