On Quantum Conditional Probability*

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ABSTRACT: We argue that quantum theory does not allow for a generalization of the notion of classical conditional probability by showing that the probability defined by the Lüders rule, standardly interpreted in the literature as the quantum-mechanical conditionalization rule, cannot be interpreted as such.

Keywords: quantum mechanics; quantum probability; conditional probability; Lüders rule; projection postulate; conceptual change

RESUMEN: Argumentamos que la teoría cuántica no admite una generalización de la noción clásica de probabilidad condicionada. Mostramos que la probabilidad definida por la regla de Lüders, interpretada generalmente como la regla de condicionalización mecánico-cuántica, no puede ser interpretada como tal.

Palabras clave: mecánica cuántica; probabilidad cuántica; probabilidad condicionada; regla de Lüders; postulado de proyección; cambio conceptual

"All the paradoxes of quantum theory arise from the implicit or explicit application of Bayes' axiom [...] to the statistical data of quantum theory. This application being unjustified both physically and mathematically" (Accardi 1984a, 298-299)

1. Introduction

Science contains many examples of concepts originating in one context and then being extended to others. Because quite distinct concepts can have the same reductions in restricted domains, however, determining what counts as an appropriate extension is a tricky business. That is all the more so because there may be no good extension at all in the new context. (Think of 'force' in special relativity or 'energy' in general relativity.) To sort out what constitutes an appropriate extension we need to be guided by certain formal features while

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also recognizing that some others may have to be let go. (Think, for example, of the idea of cardinality as extending the 'number of' concept to infinite domain, where being a proper subset no longer implies having fewer elements.) But formal features alone can never justify the extension. We also need to consider issues of interpretation; including inferential or explanatory role, and manner of application to cases.

In this paper we examine a particular question of concept extension; namely, whether the classical notion of conditional probability can be extended to the quantum domain. It is well-known that, because of its non-commutative structure, quantum mechanics does not assign joint probabilities to all pairs of quantum events; that it does so for compatible ones, but not for incompatible ones. Hence, given that conditional probability is standardly defined as the pro rata increase of a joint probability, the question arises as to whether one can introduce an appropriate notion of conditional probability at all in quantum mechanics.

A long-standing literature claims that the answer is 'yes'; that it is in fact possible to define an appropriate quantum extension of conditional probability; namely, the probability defined by the so-called 'Lüders rule'. In the context of quantum probability theory this rule satisfies the formal condition of specifying the only probability measure on the state space that reduces to a pro rata conditional probability for compatible events. Moreover, this formal condition is analogous to a similar uniqueness property of classical conditional probability. Thus, several authors have argued for interpreting the Lüders probabilities as defining the notion of conditional probability in quantum mechanics.[1] These authors claim that

> [The Lüders rule] acquires a precise meaning, in the sense of conditional probabilities, when quantum mechanics is interpreted as a generalized probability space (Cassinelli and Zanghí 1983, p.245)

> [T]he Lüders rule is to be understood as the quantum mechanical rule for conditionalizing an initial probability assignment [...] with respect to an element in the non-Boolean possibility structure of the theory (Bub 1979b, p.218)

This view is standard in the quantum mechanical literature and is so presented in a 2009 compendium of quantum physics:

> The Lüders rule is directly related to the notion of conditional probability in quantum mechanics, conditioning with respect to a single event. (Busch and Lahti 2009, p.356)

[1] Explicit arguments for this view are found in Bub (1974) and in Cassinelli and Truni (1979), which have then been expounded in Cassinelli and Zanghí (1983), Cassinelli and Zanghí (1984), Bub (1979a), Bub (1979b) and Beltrametti and Cassinelli (1981). Modern textbooks in the Philosophy of Quantum Mechanics presenting this view are, among others, (Hughes 1989) and (Dickson 1998). In addition, in (Gudder 1979) Gudder presents it as the standard view for 'quantum conditional expectation' with references that go back to at least to 1954 (Umegaki 1954).

A further rationale for understanding the probability defined by the Lüders rule in terms of conditional probability appears in the orthodox interpretation of quantum mechanics. Here the Lüders rule appears as the generalized version for degenerate eigenvalues of the so-called 'von Neumann Projection Postulate' (von Neumann 1932). It determines uniquely the state of the system after a measurement of a quantity with a given result; the new density matrix can then be used to calculate probability assignments for subsequent measurements and thus it seemingly becomes meaningful to speak of the probability distribution of a physical quantity given the result of a previous measurement of another physical quantity.

We argue against both interpretations of the probability defined by the Lüders rule as a conditional probability. First, in the context of quantum probability theory, we show that the formal analogy provided by the uniqueness result is not sufficient to ground such an interpretation. For a notion of conditional probability crucially relies on a notion of commonality between conditioned and conditioning events which cannot be found in the quantum domain in connection with the Lüders rule when incompatible events – the distinctively quantum events – are involved. And second, we show that the same difficulties appear if one interprets the probability defined by the Lüders rule in the orthodox account as a conditional-on-measurement-outcome probability beyond a merely instrumental reading.

This paper is organized as follows. In section 2, we present the main arguments in support of interpreting the Lüders rule as yielding the appropriate notion of conditional probability in quantum theory. In section 3, we argue that any well-defined notion of conditional probability should be grounded in a notion of commonality between conditioned and conditioning events. Then, in section 4, showing that quantum theory lacks any such notion in connection with the Lüders rule, we conclude that quantum theory does not allow for a generalization of classical conditional probability. Finally, in Section 5, we argue that the conditional-on-measurement-outcome interpretation is also inadequate. We offer some concluding remarks in section 6.

2. The Lüders Rule

In classical probability theory the probability of an event A conditional on another event B, $p_B(A) = p(A|B)$, is standardly defined by the ratio of two unconditional probabilities, namely the probability of their joint event $A \cap B$, $p(A \cap B)$, divided by the probability of the conditioning event B, $p(B)$, i.e.

$$p(A|B) = \frac{p(A \cap B)}{p(B)} \tag{1}$$

Now because of its non-commutative structure, quantum mechanics does not assign joint probabilities to all pairs of quantum events[2] and, hence, the ratio analysis cannot provide an analysis of conditional probability in quantum theory. Can appropriate notion of conditional probability be introduced in quantum mechanics?

At first sight one might be tempted to define the conditional probability function in a quantum probability space for two quantum projection operators P and Q in strict analogy with the classical case as $\frac{p_W(P \wedge Q)}{p_W(Q)}$, where P and Q would be, respectively, the conditioned and conditioning event, W is a density operator defining the probability functions over the quantum event structure $\mathcal{L}(\mathcal{H})$ (where $\mathcal{H}$ is the Hilbert space associated with the physical system) via the trace rule, and $\wedge$ stands for the algebraic relation meet in $\mathcal{L}(\mathcal{H})$.[3] However, the function thus defined is not a probability measure on the quantum event space $\mathcal{L}(\mathcal{H})$ since it is not an additive function. The non-additivity problem arises because the conditioning event Q need not be orthogonal to two other events P_1 and P_2 such that the conditioned event P is $P_1 + P_2$; and thus, the differences that exist between Boolean and non-Boolean event structures arise.[4] Hence, to define a conditional probability function in quantum theory, one needs, to begin with, a function linking events P and Q in an additive way.

An existence and uniqueness characterization of classical conditional probability provides the key for finding this function. Indeed, the classical conditional measure defined by the ratio $p(A \cap B)/p(B)$ is characterized as the only probability measure defined on the whole classical event space such that for events A contained in B, conditionalizing on B just involves a renormalization of the initial probability measure.[5] Analogously, start by defining in a quantum probability space a conditional probability function $m_{p_W}(P) = \frac{p_W(P)}{p_W(Q)}$ for projectors $P \leqslant Q$ (since the sub-lattice of projectors $P \leqslant Q$, i.e. $\mathcal{L}(Q)$, is Boolean, this function is defined analogously to the classical one), then, similarly to the classical case, the function $m_{p_W}(\cdot)$ defined in $\mathcal{L}(Q)$ can be extended to all $\mathcal{L}(\mathcal{H})$ in a unique way.[6]

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[2] The relation between the non-existence of the joint distribution of two observables and their incompatibility is subtle and depends critically on the fact that a joint distribution is defined in terms of a particular state W. See Gudder (1968), Gudder (1979), Malley (2004), Malley and Fine (2005), Malley (2006), Malley and Fletcher (2008), Nelson (1967) and Varadarajan (1962).

[3] For a detailed formulation of quantum probability theory see, for example, Beltrametti and Cassinelli (1981), Bub (1974) or Hughes (1989).

[4] The fact that $\mathcal{L}(\mathcal{H})$ is not distributive is precisely what precludes the function from being a probability function. Indeed, if Q is not orthogonal to two orthogonal events P_1 and P_2, then $(P_1 + P_2) \wedge Q \neq P_1 \wedge Q + P_2 \wedge Q$, and hence $p_W[(P_1 + P_2) \wedge Q] \neq p_W(P_1 \wedge Q) + p_W(P_2 \wedge Q)$ which then implies non-additivity of the probabilities defined by $\frac{p_W(P \wedge Q)}{p_W(Q)}$.

[5] See Cassinelli and Zanghí (1983), Teller and Fine (1975), Hughes (1989).

[6] See Beltrametti and Cassinelli (1981, p.288), Cassinelli and Zanghí (1983), Malley (2004,

Theorem 1. Existence and Uniqueness.

Let Q be any projector in the lattice $\mathcal{L}(\mathcal{H})$ of projectors of a Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) \geqslant 3$. Let $p(\cdot)$ be any probability measure on $\mathcal{L}(\mathcal{H})$, with corresponding density operator W, such that $p_W(Q) \neq 0$. For any P in $\mathcal{L}(Q)$ define $m_{p_W}(P) = \frac{p_W(P)}{p_W(Q)}$, where $p_W(P) = \mathrm{Tr}(WP)$, as fixed by Gleason's theorem. Then,

1. *$m_{p_W}(\cdot)$ is a probability measure on $\mathcal{L}(Q)$*
2. *there is an extension $p_W(\cdot|Q)$ of $m_{p_W}(\cdot)$ to all $\mathcal{L}(\mathcal{H})$*
3. *the extended probability measure $p_W(\cdot|Q)$ is unique and, for all P in $\mathcal{L}(\mathcal{H})$, is given by the density operator $W_Q = \frac{QWQ}{\mathrm{Tr}(QWQ)}$ so that*

$$p_W(P|Q) = p_{W_Q}(P) = \mathrm{Tr}\left(W_Q P\right) = \frac{\mathrm{Tr}(QWQP)}{\mathrm{Tr}(QWQ)} \tag{2}$$

Expression (2) is referred to as the Lüders rule. For a system in a pure state represented by the vector ψ, it is rewritten as

$$p_\psi(P|Q) = p_{\psi_Q}(P) = \langle \psi_Q, P\psi_Q \rangle \tag{3}$$

where $\psi_Q = \frac{Q\psi}{\|Q\psi\|}$.

The formal result of theorem 1 is standardly invoked to support an interpretation of the Lüders rule as defining the appropriate notion of conditional probability on the quantum event structure $\mathcal{L}(\mathcal{H})$. The reasoning given is as follows:

> [...] as in the classical case, the Lüders rule gives the only probability measure that, for events $P \leqslant Q$, just involves a renormalization of the [initial] generalized probability function [p_W] given by the operator W. This offers strong grounds for regarding it as the appropriate conditionalization rule for generalized probability functions on $\mathcal{L}(\mathcal{H})$ (Hughes 1989, p.224, notation adapted).

In addition, the probabilities defined by the Lüders rule reduce to classical conditional probabilities for compatible events.[7] Hence the claim is that the Lüders rule

> is the appropriate rule for conditionalizing probabilities in the non-Boolean possibility structure of quantum mechanics. (Bub 1977, p.381)

pp.13-15).

[7] If events P and Q are compatible then the corresponding projection operators commute so that $PQ = QP = R$, where R projects onto the intersection of the subspaces associated with P and Q. Inserting this into the Lüders rule, and using the invariance of the cyclic permutations of the trace operation, one obtains that $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(QWQP)}{\mathrm{Tr}(QWQ)} = \frac{\mathrm{Tr}(WR)}{\mathrm{Tr}(WQ)} = \frac{p(R)}{p(Q)} = \frac{p(L_P \cap L_Q)}{p(L_Q)}$.

Now classical conditional probability, in addition to being characterized by its existence and uniqueness property, is also characterized by being the only measure which is necessarily additive with respect to conditioning events. It turns out that the probability defined by the Lüders rule lacks this additive property and it is precisely because of this that the Lüders rule can account for the specifically quantum interference effects and match the quantum statistics.[8] Thus Cassinelli and Zanghí write:

> [...] the generalized conditional probability maintains all the characterizing features of the classical one and, at the same time, it introduces typical quantum effects. The essential point is that, in the non-commutative case the *theorem of compound probabilities* [or, equivalently, additivity with respect to conditioning events,] does not hold (Cassinelli and Zanghí 1984, p.244)

The quantum notion agrees with its classical counterpart when it applies to compatible events but differs from it when incompatible events are involved. In these cases it cannot be interpreted as a classical conditional probability but rather is seen as providing an extension of this notion appropriate for the quantum context.

3. Commonality and Conditional Probability

The probability defined by the Lüders rule reduces to a classical conditional probability when the events involved are compatible. As such, it is able to capture some of the basic intuitions or essential features of the notion of conditional probability. For example, the probability defined by the Lüders rule for $P = Q$, i.e. $p_W(P|P)$, is one, thus preserving the intuition that the probability of any event P given itself is one; or $p_W(P^{\perp}|P) = 0$, complying with the fact that the probability of the complement of any event given the event itself must be zero. The crucial question, however, is whether this probability can be understood as an extension of classical conditional probability to the quantum context, that is, for incompatible events which are the distinctively quantum events.

To illustrate this question, consider a spin $\frac{1}{2}$ particle in a state corresponding to a positive value of spin along the z-axis, i.e. $\psi_{s_{+z}}$. The probability for the event $P = P_{s_{+z}}$ corresponding to a positive value of spin along the z-axis in this state is one. What is the probability for this same event conditional on another event Q such that $p_W(Q) \neq 0$, say $Q = P_\phi$, with $\phi = a\psi_{s_{+z}} + b\psi_{s_{-z}}$ ($|a|^2 + |b|^2 = 1$),[9] a linear combination of the event corresponding to a positive and a negative value of spin along the z-axis?

According to what we know about classical conditional probability, one might begin by considering the two following intuitions. On the one hand, the

[8] See Cassinelli and Zanghí (1984). See also Beltrametti and Cassinelli (1981, pp. 281-285).

[9] For $Q = P_\phi$, $\phi = a\psi_{s_{+z}} + b\psi_{s_{-z}}$ and $\psi = \psi_{s_{+z}}$, $p_\psi(Q) = |\langle\phi|\psi_{s_{+z}}\rangle|^2 = |a|^2$. Hence, $p_\psi(Q) \neq 0$ if $a \neq 0$.

probability of $P_{s_{+z}}$ conditional on P_ϕ should be equal to one for, given that the unconditional probability of $P_{s_{+z}}$ is already one, considering any other event whose probability is not zero should leave this value unaltered. On the other hand, the probability of $P_{s_{+z}}$ conditional on P_ϕ should seemingly not take any value different from zero, be it one or any other value. For $P_{s_{+z}}$ and P_ϕ seem to have nothing in common since the intersection of their ranges is zero. And hence the conditional probability of $P_{s_{+z}}$ given P_ϕ should be zero. However, the probability $p_{\psi_{s_{+z}}}(P_{s_{+z}}|P_\phi)$ defined by the Lüders rule is not generally assigned either the value one or zero; rather its value can range from 0 to 1 depending on the value of a given that $p_{\psi_{s_{+z}}}(P_{s_{+z}}|P_\phi) = |a|^2$.[10] How can we then understand the probability $p_{\psi_{s_{+z}}}(P_{s_{+z}}|P_\phi)$ as the probability of the event P_{s+z} informed or qualified by the event P_ϕ?

The notion of conditional probability is that of the probability of an event – the conditioned event – informed or qualified by the occurrence of another event – the conditioning event. In classical probability theory it is easy to understand why the ratio formula (1) can capture this notion.[11] Indeed, imagine a fair die is about to be tossed. What is the probability that it lands with '1' showing up conditional on or given that the outcome is an odd number?

First, if we know that the outcome of the throw is an odd number, then the appropriate sample space for calculating the probability of '1' is not $S = \{1, 2, 3, 4, 5, 6\}$ anymore; rather S gets replaced by the set of odd outcomes $S_{\text{odd}} = \{1, 3, 5\}$. Second, there are many probability measures on this new sample space; what specifically defines the conditional probability measure is that the sample space changes from S to S_{odd} and *nothing else*. That is, the conditional probability given 'odd', by definition, differs from the original one solely by taking into account the qualification of an odd outcome. This means that one has to eliminate the points in S that are not in S_{odd} (2, 4 and 6), without altering the relative probability of the points which remain (1, 3 and 5), i.e. by increasing the latter's value 'pro rata'. Thus, $\mathbb{P}_p(1|\,\text{odd})$ is derived from the initial probability measure by dividing the initial measure by the initial probability of odd, i.e. $\mathbb{P}_p(1|\,\text{odd}) = \frac{p(1)}{p(\text{odd})} = \frac{1/6}{1/2} = \frac{1}{3}$.

In this example $A = \{1\}$ is a subset of $B = \{\text{odd}\}$; for general subsets A that are not necessarily subsets of B, as for example $A = \{1, 2, 3\}$ and $B = \{\text{odd}\}$, one has to consider only the probability of the sample points in A that are also in B and disregard the rest. For the sample points in A that are not also in B will not be possible outcomes in the new event space S_B and, therefore, will be assigned zero probability. Thus, in general, the conditional probability measure on B is generated by looking at the probability of the intersection $A \cap B$ of any measurable A with B, and increasing it pro rata

[10] Given $\psi_Q = \frac{Q\psi}{\|Q\psi\|} = \frac{P_\phi \psi_{s_{+z}}}{\|P_\phi \psi_{s_{+z}}\|} = \phi$, $\mathbb{P}_\psi(P|Q) = \langle \psi_Q, P\psi_Q\rangle = \langle \phi, P_{s_{+z}}\phi\rangle = |\langle \phi|\psi_{s_{+z}}\rangle|^2 = |a|^2$.

[11] We do not consider the difficulties that appear in interpreting the ratio formula as a classical conditional probability. See Hájek (2003), Hájek (2009).

(so that the new conditional measure remains normalized). For the notion of classical conditional probability the relation of commonality between the two events is, hence, the crucial aspect; it determines how the sample space changes and what events one has to consider. And then a pro rata adjustment of the initial probability function suffices.

More generally, a relation of commonality between the conditioning and the conditioned events seems to be essential to any notion of conditional probability. It does not have to, as in the classical case, be captured by looking at the subset intersection of the two events; but some kind of link between the conditioning and conditioned events seems necessary if the occurrence of the former is to determine how the probability of the latter changes.[12] Is there then some quantum notion of commonality between the events $P_{s_{+z}}$ and P_ϕ to ground thinking of the probability defined by the Lüders rule $\mathbb{P}_{\psi_{s_{+z}}}(P_{s_{+z}}|P_\phi)$ as a conditional one?

4. No Quantum Notion of Commonality

If the probability defined by the Lüders rule is to be understood as a conditional probability for incompatible quantum events, it cannot rely on the classical notion of commonality; that is, one cannot think of a notion of commonality between projectors in terms of the intersection of their subspaces. Rather, one needs a notion of commonality which can first, cope with the fact that $\mathbb{P}_W(P|Q)$ is in general non-zero for events P and Q such that the intersection of their ranges is zero, and second, determine the particular non-zero value which the Lüders rule actually assigns $\mathbb{P}_W(P|Q)$.[13]

A look at the probability defined by the Lüders rule quickly suggests the following one. By the invariance of the trace under cyclic permutations, this

[12] I take as a starting point, without giving further arguments, that if the occurrence of an event is to determine how the probability of another changes then some kind of link between both events seems necessary. (A similar situation occurs when discussing the notion of negation in the Appendix.) The difficulty lies in evaluating what constitutes a proper conceptual extension. Indeed, for a notion to be the conceptual extension of another there needs to be some 'core' meaning which both concepts share and carries over the boundary between the old and the new theoretical context. However, in general, the concept of the new theory will have some features in common with the old concept and some completely new features. And there do not seem to be clear cut criteria to determine which of the properties of the old concept the new concept should necessarily retain to regard it as an extension of the old one rather than a different concept altogether. I take it that commonality *is* a core feature of the concept of conditional probability.

[13] The fact that a notion of commonality based on subspace intersection cannot underwrite a quantum notion of conditional probability is easily understood. Indeed, this notion corresponds to defining conditional probability by the ratio $\frac{p_W(P \wedge Q)}{p_W(Q)}$, which we showed does not define a probability function over $\mathcal{L}(\mathcal{H})$.

probability can be written as $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(QWQP)}{\mathrm{Tr}(QWQ)} = \frac{\mathrm{Tr}(WQPQ)}{\mathrm{Tr}(QWQ)}$. If P is a one-dimensional projection operator onto the ray α, i.e. $P = P_\alpha$, one can write $QP_\alpha Q$ as $P_{Q\alpha}$,[14] i.e. the projector onto the (non-normalized) vector $Q\alpha$, thus obtaining $\mathbb{P}_W(P_\alpha|Q) = \frac{\mathrm{Tr}(WP_{Q\alpha})}{p_W(Q)} = \frac{p_W(P_{Q\alpha})}{p_W(Q)}$. In analogue fashion to the classical process for generating the conditional probability measure on B by looking at the probability of the intersection $A \cap B$ of any measurable A with B, and increasing it pro rata, one could argue that in quantum theory the projector $P_{Q\alpha}$ represents the common quantum projector of Q and P_α which is assigned a probability by means of the standard trace rule, and then is increased pro rata. If P is not one-dimensional, then it is the effect operator QPQ which represents the common projector of P and Q.

One would then, seemingly, not only explain why one should assign a non-zero probability to projector P conditional on projector Q when $P \wedge Q = 0$, but also why it takes the particular non-zero value the Lüders rule assigns it. Indeed, for the spin $\frac{1}{2}$ example even if the intersection of the ranges of $P_{s_{+z}}$ and P_ϕ, where $\phi = a\psi_{s_{+z}} + b\psi_{s_{-z}}$, is zero, i.e. $P_{s_{+z}} \wedge P_\phi = 0$, their common projector is given by $PQP = P_{Q\alpha} = P_{a^*\phi}$, and thus probability of this common projector increased pro rata yields $\mathbb{P}_{\psi_{s_{+z}}}(P_{s_{+z}}|P_\phi) = \frac{p_{\psi_{s_{+z}}}(P_{a^*\phi})}{p_{\psi_{s_{+z}}}(P_\phi)} = |a|^2$. Hence, by appealing to a notion of commonality of quantum projectors based on their projective geometry, it looks like the probability defined by the Lüders rule could be interpreted as a conditional probability.

This projective reading, however, only holds at a superficial level. First, it seems counterintuitive to regard QPQ as the common projector of Q and P. For in $\mathbb{P}_W(P|Q)$ the common projector is the operator QPQ, while in $\mathbb{P}_W(Q|P)$ it is PQP, which are in general different from each other. And yet, why should they be different if they are both supposed to represent what P and Q have in common? One could bite the bullet and simply stipulate that QPQ is by definition the common projector of P and Q in the projective lattice $\mathcal{L}(\mathcal{H})$, and argue that our intuitions about what is counterintuitive are not reliable; for these are molded on Boolean structures and we are considering projection operators which precisely have a non-Boolean structure. These responses, however, only make the understanding of the probability defined by the Lüders rule as a conditional probability too superficial to be considered as such given that they do not really engage the interpretive question. They provide no understanding of why the common projector of P and Q is QPQ, nor of why the common quantum projector for the probability of P conditional on Q is different form that of Q conditional on P.

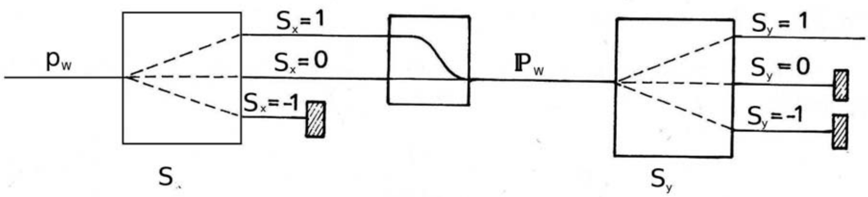
Moreover, important difficulties appear as soon as one considers whether this understanding in terms of projection operators can provide a rationale for thinking of the probability defined by the Lüders rule as a conditional probability for the values of physical quantities of a quantum system. Indeed,

[14] For any vector ϕ, $(QP_\alpha Q)\phi = QP_\alpha(Q\phi) = Q\alpha\langle\alpha, Q\phi\rangle = Q\alpha\langle Q\alpha, \phi\rangle = P_{Q\alpha}\phi$.

it applies directly to projection operators on a Hilbert space $\mathcal{H}$ but these are only physically meaningful through their associated eigenvalues. Hence, the projective reading would be an adequate interpretation of the probability defined by the Lüders rule only in so far as it could underwrite a quantum notion of conditional probability in terms of physically relevant values. That is, a reading of $\mathbb{P}_W(P|Q)$ as the probability for value p – the eigenvalue associated with P – conditional on value q – the eigenvalue associated with Q.

Now to ground this notion of quantum conditional probability one would need to find a notion of commonality in terms of the physical values p and q that would somehow correspond to the operator QPQ. The problem is, however, two-fold. First, there are many cases in which there simply seems to be no understanding of the conjunction of two physical values p and q of a quantum system. And second, in those cases in which the event 'p and q' can be represented in terms of the projection operators P and Q, it is represented by $P \wedge Q$ rather than by QPQ, and neither these operators nor the probability assigned to them are in general equal. Only if P and Q are compatible projectors do all these notions line up – 'p and q' can be represented by the projector $P \wedge Q$, which in turn is equivalent to the common operator QPQ – and the probability $\mathbb{P}_W(P|Q)$ can be read as the pro rata increase of 'p and q'.

Begin with the first problem. The difficulty here is directly related to those quantum scenarios in which interference effects are present; these arise when the 'conditioning' event Q is not drawn back to the occurrence of the single events Q_i that compose it. For example, consider a spin-1 particle and two Stern-Gerlach devices that separate the possible values of the spin component, viz. $-1, 0, 1$, along the x- and y- axis (Beltrametti and Cassinelli 1981, pp. 281-285).



Let Q_1 and Q_2 be the events 'the x-component is $+1$' and 'the x-component is 0', respectively, i.e. $Q_1 = P_{s_{x_{+1}}}$ and $Q_2 = P_{s_{x_0}}$; and let P be the event 'the y-component is $+1$', i.e. $P = P_{s_{y_{+1}}}$. The probability defined by the Lüders rule for $P = P_{s_{y_{+1}}}$ and $Q = P_{s_{x_{+1}}} + P_{s_{x_0}}$, i.e. $\mathbb{P}_\psi(P_{s_{y_{+1}}}|P_{s_{x_{+1}}} + P_{s_{x_0}})$, agrees with the empirical probability for finding the y-component of spin to be $+1$ in the experiment shown in figure 1. The question is whether this probability can be interpreted as the probability that the y-component of spin is $+1$ when informed or qualified by the x-component of spin being $+1$ or 0. As we have argued, for this to be so we need to find an understanding of the conjunction of physical values $s_{y_{+1}}$ and $s_{x_{+1}}$ or s_{x_0}, whose probability increased pro rata yields such a notion of conditional probability.

In the quantum event structure $\mathcal{L}(\mathcal{H})$ one can define algebraic relations between the projection operators representing quantum events which are the counterparts of set-union, set-intersection, and set-complementation. These are, respectively, $P \vee Q$, $P \wedge Q$, and $P^{\perp}$. And given that in classical logic the logical relations between events correspond naturally to the algebraic relations between the subsets that represent those events – A *or* B is represented by $A \cup B$; A *and* B is represented by $(A \cap B)$; and *not* A is represented by A^c – the straightforward suggestion is, hence, that the algebraic relations in $\mathcal{L}(\mathcal{H})$ correspond to 'quantum logical relations': '$\vee$' is to be interpreted as providing an extension of the classical *or* in the quantum domain, '$\wedge$' as the quantum logical *and*, and '$\perp$' as the quantum logical 'not'. Thus, the event the y-component of spin is +1 and the x-component is +1 or 0 would be represented by the projector $P_{s_{y_{+1}}} \wedge (P_{s_{x_{+1}}} \vee P_{s_{x_0}})$.

However, the analogy between the correspondence of the algebraic and logical relations in classical and quantum mechanics turns out to hold only at a purely formal level, and thus the algebraic relations '$\vee$' and '$\perp$' of $L(\mathcal{H})$ cannot be understood, respectively, as generalized or extended notions of the ordinary logical concepts of disjunction and negation. Briefly, the difficulty is that the operator '$\perp$' cannot be interpreted as logical negation or as an extension of it. And this, in turn, precludes understanding '$\vee$' as the quantum extension of ordinary disjunction. (See the Appendix for a detailed argument.) In terms of our example, the problem is that $P_{s_{x_{+1}}} \vee P_{s_{x_0}}$ (which is equal to $P_{s_{x_{+1}}} + P_{s_{x_0}}$ since the projection operators are orthogonal) dose not seem to allow its interpretation as the event '$s_{x_{+1}}$ *or* s_{x_0}', and hence the projector $P_{s_{y_{+1}}} \wedge (P_{s_{x_{+1}}} \vee P_{s_{x_0}})$ cannot be read as the event '$s_{y_{+1}}$ and $s_{x_{+1}}$ or s_{x_0}'. For the truth conditions that quantum mechanics (with its reliance on the e-e link) dictates for '$\vee$' do not allow its understanding as logical disjunction or an extension of it.

Consider, for example, the assignment of truth values dictated by a general (pure) state $\psi = c_{+1}\psi_{s_{x_{+1}}} + c_0\psi_{s_{x_0}} + c_{-1}\psi_{s_{x_{-1}}}$. The disjunct $P_{s_{x_{+1}}} \vee P_{s_{x_0}}$ is true in ψ (because ψ is an eigenstate of P_I), while neither $P_{s_{x_{+1}}}$ nor $P_{s_{x_0}}$ are true in ψ (because if both c_{+1} and c_0 are different from zero, ψ is not an eigenstate of $P_{s_{x_{+1}}}$ nor of $P_{s_{x_0}}$). Hence, for the system in state ψ, S_x does *not* take value +1 *nor* does it take value 0, yet S_x *does* take some value. In other words, to read $P_{s_{x_{+1}}} \vee P_{s_{x_0}}$ as '$s_{x_{+1}}$ *or* s_{x_0}' only makes sense if we hold that the electron has an x-component of spin +1 or 0, even though it is false that it has an x-component of spin +1 and it is also false that it has an x-component of spin 0. This certainly appears to preclude understanding '$\vee$' as logical disjunction.

Moreover, even if one were to accept that the event '$s_{y_{+1}}$ and $s_{x_{+1}}$ or s_{x_0}' is somehow represented by the projector $P_{s_{y_{+1}}} \wedge (P_{s_{x_{+1}}} \vee P_{s_{x_0}})$, the probability assigned to this projector increased pro rata would still not agree with the probability dictated by the Lüders rule given that is different from the projector employed by the Lüders rule, i.e. $QPQ = (P_{s_{x_{+1}}} + P_{s_{x_0}})P_{s_{y_{+1}}}(P_{s_{x_{+1}}} + P_{s_{x_0}})$. This is the second problem we pointed to before. Hence, given that there is no notion of commonality in terms of the physical values $s_{y_{+1}}$ and $s_{x_{+1}}$ or s_{x_0} that corresponds to the operator QPQ, it seems that the probability $\mathbb{P}_\psi(P_{s_{y_{+1}}}|P_{s_{x_{+1}}} + P_{s_{x_0}})$ defined by the Lüders rule cannot be interpreted as a conditional probability for the physical values associated to these projectors.

To conclude, the fact that the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule agrees with classical conditional probability in their shared domain of application, i.e. compatible events, does not guarantee that outside this domain, i.e. incompatible events, it retains enough interpretive content so as to justify regarding it as a genuine extension of the classical notion; and this regardless of the fact that the probability defined by the Lüders rule is the only possible candidate for a quantum notion of conditional probability (recall theorem 1). Although superficially it may seem that the uniqueness argument of section 2 underwrites a quantum notion of conditional probability, we have argued that the notion of the probability of an event qualified by the occurrence of another requires a notion of commonality between events which cannot be found in the quantum domain.[15]

5. *A Rationale for Measurement Results?*

Now an orthodox interpreter of quantum theory would try to resist our conclusion by claiming that there is a notion of conditional probability in quantum mechanics if physical values are interpreted as measurement results. Indeed, the Lüders rule appears in the orthodox interpretation of quantum mechanics as the generalized version for degenerate eigenvalues of the so-called 'von Neumann Projection Postulate' (von Neumann 1932). It determines uniquely the state of the system after a measurement of a quantity with a given result; the new density matrix can then be used to calculate probability assignments for subsequent measurements and thus it seemingly becomes meaningful to speak of the probability distribution of a physical quantity given the result of a previous measurement of another physical quantity.

In more detail, imagine we perform an ideal first-kind (i.e. non-destructive and repeatable) measurement on a system in state W of a certain observable, where Q belongs to its spectral decomposition, and find measurement outcome q. According to the Lüders projection postulate the new state of the system will be given by the density operator $W_q = \frac{QWQ}{\mathrm{Tr}(QWQ)}$. If we then perform a measurement of a second observable, where P belongs to its spectral decomposition,

[15] My claim agrees with that presented by Leifer and Spekkens, namely 'the assertion that the projection postulate is analogous to Bayesian conditioning is based on a misleading analogy.' Leifer and Spekkens (2012, p.3), but our reasons are different.

the probability to find measurement outcome p in this second measurement is given by the trace rule as

$$\mathbb{P}_W(p|q) \equiv p_{W_q}(p) = \mathrm{Tr}\left(\frac{QWQ}{\mathrm{Tr}(QWQ)}P\right) \tag{4}$$

(For a system in a pure state ψ, the state of the system 'collapses' upon measurement to the state $\psi_q = \frac{Q\psi}{\|Q\psi\|}$, i.e. the normalized projection of ψ onto the eigenspace belonging to q, and (4) reads $\mathbb{P}_\psi(p|q) = p_{\psi_q}(p) = \langle \psi_q, P\psi_q \rangle$). Hence, the proposal is that the probabilities defined by the Lüders projection postulate $\mathbb{P}_W(p|q)$ define the notion of conditional probability in quantum mechanics for measurement results: the probability of measurement outcome p conditional on measurement outcome q.[16]

Now if when one says the probability of a certain outcome p of an experiment conditional on a previous measurement result q is p_{W_q} as given by the Lüders rule, one only means that if the experiment is repeated many times one expects that the fraction of those which give the outcome p when outcome q has been previously found is roughly p_{W_q}, then we have no quarrel with the conditional-on-measurement-outcome interpretation. But if one goes a step further in trying to understand p_{W_q} as a genuine conditional probability then difficulties arise. This interpretive problem is standardly accepted for primitive unconditional probabilities: if when one says the probability of a certain outcome p of an experiment is p_W as given by the trace rule, one only means that if the experiment is repeated many times one expects that the fraction of those which give the outcome p is roughly p_W, then no problems arise. But what does this probability really mean beyond this instrumental understanding? This is yet an unresolved question.[17]

For the conditional probability notion the interpretive problems are those we discussed on the previous section. Put in terms of measurement results, the fact that one learns measurement outcome q is the case is not informative of how the probability of measurement outcome p should change from p_W as given by the trace rule to p_{W_q} as given by the Lüders rule. One knows how to calculate the new probability p_{W_q} by applying the Lüders rule, but the change from p_W to p_{W_q} cannot be understood on account of the relation between measurement results p and q as it happens in the classical case. Hence, it seems that the orthodox interpreter cannot provide an adequate conditional interpretation of the probability defined by the Lüders projection postulate

[16] Note that this probability – probability for a measurement outcome given another measurement outcome – is different from the orthodox quantum probabilities for measurement outcomes conditional on measurements. The latter are conditional on the role of background conditions which specify the conditions in effect at the assessment of a probability function – in this case, the measurement procedure – while the former are conditional on specific measurement results.

[17] See for example Pitowsky (1989).

as a conditional-on-measurement-outcome probability over and above a purely instrumental reading.

6. *Conclusion*

Extending concepts into new domains is always a tricky business. As we have seen, it is not sufficient to show that there are some formal analogies between the old and the extended concept. In addition, it is essential to evaluate whether these analogies can provide enough interpretive content so as to justify regarding the concept in the new domain as an extension or a generalization of the old one.

Trying to extend the concept of conditional probability in the probabilistic framework of the quantum theory is especially tricky. In this case, formal analogies, which suggest the probabilities defined by the Lüders rule as the natural extensions, appear to not provide enough interpretive content so as to justify regarding them as extensions of classical conditional probabilities to the quantum context from a meaningful perspective. It seems doubtful whether there is any natural extension of conditionality that fits the non-commutative structure of the quantum theory – a domain where joint events make no sense – and illuminates how it is applied. Even though there exists a well-defined notion of conditional probability for compatible quantum events which can be extended uniquely to apply to general quantum events – the so-called Lüders rule – we have argued that this extension is merely formal and does not carry the required interpretative weight.

7. *Appendix: Interpreting Quantum Logic*

Our conclusion in section 4 that the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be read as the probability of physical value p conditional on physical value q (where p and q are the eigenvalues associated with projectors P and Q) relied on two claims, namely that (1) in those scenarios in which interference effects are present, the projector $Q = Q_1 \vee Q_2$ cannot be interpreted as the event 'q_1 *or* q_2', and thus $P \wedge Q$ cannot be understood as 'p *and* q_1 *or* q_2'; and (2) in those cases in which the event 'p and q' can be represented in terms of the projection operators P and Q, it is represented by $P \wedge Q$ rather than by QPQ, and neither these operators nor the probability assigned to them are in general equal. In this appendix we provide a detailed defense of the first claim by showing why disjunction '$\vee$' cannot be understood as an extension of the logical relation 'or'.

As we saw in section 4, in the quantum event structure $\mathcal{L}(\mathcal{H})$ one can define algebraic relations between the projection operators representing quantum events which are the counterparts of set-union, set-intersection, and set-complementation. Many of the relations between the quantum algebraic relations are similar to those between the classical ones. For example, just as

$(A^c)^c = A$ holds for set-complementation, $(P^{\perp})^{\perp} = P$ holds for subspace-complementation; or just as $A \cup A^c = S$ in $\mathcal{F}(S)$, $P \vee P^{\perp} = I$ in $\mathcal{L}(\mathcal{H})$; or similarly to $(A \cap A^c)^c = S$, we have $(P \wedge P^{\perp})^{\perp} = I$.

Now, in classical logic, the logical relations between events (or propositions representing those events) correspond naturally to the algebraic relations between the subsets that represent those events. The suggestion is, thus, that, given the similarities between the classical and the quantum algebraic relations, the algebraic relations in $\mathcal{L}(\mathcal{H})$ correspond to 'quantum logical relations', where it is assumed that these provide some kind of extension of our ordinary logical notions in the quantum context. Thus, the algebraic relation '$\vee$' between quantum projectors is interpreted as the quantum logical 'or' for quantum events. Similarly, the algebraic relation '$\wedge$' is taken to correspond to the quantum logical 'and', and '$\perp$' is read as the quantum logical 'not'.

However, just as with conditional probability, these analogies turn out to hold only at a purely formal level. That is, the algebraic relations '$\vee$' and '$\perp$' for quantum projectors cannot be understood, respectively, as generalized or extended notions of the ordinary logical concepts of disjunction and negation from any physically meaningful perspective. And hence, for example, even if formally one has $(P^{\perp})^{\perp} = P$, one should not interpret this equation as double negation; nor should one interpret $P \vee P^{\perp}$ or $(P \wedge P^{\perp})^{\perp}$ as the logical laws of excluded middle and non-contradiction respectively.

Fine (1972) argues for this conclusion by constructing the analogue to quantum logic for a simple, two-dimensional system. In this logic it is clear that the meaning of the algebraic relations differs substantially from the meaning of the ordinary logical relations. And hence, the conclusion that the former cannot be regarded as extensions of the latter. These conclusions then carry over to quantum logic. Let us first consider this toy model and then look at quantum logic.

7.1. A Toy Model

Consider the location of a certain point P on a given circle C. Suppose that for the location of P there are three accessible regions: (1) the center of the circle, (2) the entire area of the circle and (3) any diameter of the circle. Any sentence of the form 'P is on X', where X is one of the accessible regions, corresponds to an elementary sentence. The idea is to construct a logic from the elementary sentences by introducing sentential connectives and truth conditions.

Let us first introduce the binary connective '$\wedge$' such that for elementary sentences 'P is on X' and 'P is on Y', the conjunction

'P is on X' $\wedge$ 'P is on Y' $\equiv$ 'P is on Z'

where Z describes the region of the circle that is the intersection of the X and Y regions. One can readily verify that the intersection of two accessible regions is again an accessible region and, therefore, that conjunction is well-defined. The functor '$\wedge$' is just the usual sentential conjunction with regard to the interpretation of sentences as locating the particle on the circle.

It is also the usual conjunction with regard to truth conditions. In effect, each possible location L for the particle P that is on the circle but not at the center yields an assignment of truth values according to the prescription 'P is on X' is true under L iff under L, P is on X. Hence the sentence 'P is on the center of the circle' is false under all truth assignments (and will play the role of 'the false' in this system.) And the semantic rule for conjunction is thus defined as follows. If ϕ, ψ are elementary sentences, an assignment L of truth values to the elementary sentences automatically assigns truth values to conjunctions according to the rule '$\phi \wedge \psi$' is true under L iff 'ϕ' and 'ψ' are true under L. The functor '$\wedge$' is thus also the usual conjunction with regard to truth conditions.[18]

The situation with negation is, however, quite different. If one wanted to introduce the usual negation, then one should introduce a unary functor '$\sim$' as

$\sim$ (P is on X) $\equiv$ P is in the circle but not on the regions described by X

The problem with the '$\sim$' definition of negation is that the set of elementary sentences is not closed under it. For example, if X describes a diameter, then $\sim$ (P is on X) describes the circle minus a diameter, which is not an accessible region. For the elementary sentences to be closed under negation '$\sim$' one can either expand the list of accessible regions so as to include with each region on the list its complement relative to the circle (and then introduce ordinary negation as above), or retain the previous list of accessible regions by introducing a unary functor under which the elementary sentences are closed. The new functor will, therefore, be different from the ordinary sentential negation.

Consider the second option and define the unary functor '$\neg$' as

$\neg$(P is on X) $\equiv$ P is on $X^{\perp}$

where if R is the region described by X, then $X^{\perp}$ describes (1) the center of the circle if R is the whole circle, (2) the whole circle if R is the center of the circle, and (3) the diameter perpendicular to R if R is a diameter. Note that '$\neg$' satisfies the desired involutary property, namely $\neg(\neg P) = P$. Also, an assignment L of truth values to the elementary sentences automatically assigns truth values to $\neg\phi$ according to the rule 'if 'ϕ' is true under L, then '$\neg\phi$' is false under L'. However, contrary to conjunction, the functor '$\neg$' is *not* the usual logical negation, both with regard to the interpretation of sentences as locating the particle on the circle and with regard to truth conditions.

[18] Note that the semantic notions of validity and logical equivalence are defined as usual, namely ϕ is valid iff ϕ is true under all assignments of truth values, and ϕ is logically equivalent to ψ iff ϕ and ψ have the same truth value under all assignments.

First, to deny that point P is in diameter X is not to assert that it is in the diameter perpendicular to X, as 'negation' $\neg$ prescribes. Indeed, the point could be anywhere in the circle! Second, whereas the above semantic rule holds for 'negation' '$\neg$', its converse – while true for ordinary negation – does not hold here. For example, suppose that the assignment L derives from P being on diameter X. If ϕ is the sentence 'P is on Y', where Y describes a diameter not perpendicular to the X diameter, then both 'P is on Y' and 'P is on $Y^{\perp}$' are false under L; that is, both 'ϕ' and '$\neg\phi$' are false under L. The trouble arises because if it is false that P is on a certain diameter, it does not follow that P is on the perpendicular diameter. Thus, even though the set of elementary sentences is closed under functor '$\neg$', it is *not* ordinary negation nor an extension of it.

Finally, given conjunction and negation, one can introduce disjunction by the De Morgan Laws $(\phi \vee \psi) \equiv \neg(\neg\phi \wedge \neg\psi)$. The semantics forced on disjunction by this definition are as follows:

If 'ϕ' is true under L or 'ψ' is true under L, then '$\phi \vee \psi$' is true under L

The converse, however, does not hold, that is, the disjunction can be true although neither disjunct is true. For example, if ϕ, ψ locate P in distinct diameters, then the disjunction $(\phi \vee \psi)$ is true under all assignments of truth values, since it merely says that P is somewhere on the circle. And under an assignment in which P is on neither of the mentioned diameters, each disjunct will be false, whereas the disjunction as a whole will be true. Also notice that if ϕ, ψ locate P in distinct diameters, the conjunction $(\phi \wedge \psi)$ is false under all assignments, since it would place P on the center of the circle. Thus, the algebraic relation '$\vee$' cannot be understood as disjunction nor as an extension of it.

To finish, let us look at the distributive law. Suppose ϕ_1, ϕ_2, ϕ_3 locate P on distinct diameters R_1, R_2, R_3 respectively. The conjunction $(\phi_1 \vee \phi_2) \wedge \phi_3$ locates P on R_3 while the disjunction $(\phi_1 \wedge \phi_3) \vee (\phi_2 \wedge \phi_3)$ locates P on the center of the circle. Thus the latter disjunction is false under every assignment of truth values while the former conjunction is true under the assignment where P is on R_3. Hence $(\phi_1 \vee \phi_2) \wedge \phi_3 \neq (\phi_1 \wedge \phi_3) \vee (\phi_2 \wedge \phi_3)$, and the distributive law does not hold in this 'circular logic'.

The distributive law fails due to the oddities of disjunction, which in turn derive from the nonstandard 'negation' '$\neg$'. But given that the latter differs in meaning (with regard to both interpretation and truth conditions) form ordinary negation, the failure of the distributive law for this system does not illustrate how the ordinary law of distributivity might be false. To assert the distributive law in this circular logic is not to assert the ordinary distributive law at all.

7.2. Quantum Logic

Similarly to the definition of 'negation' '$\neg$' in the 'circular logic', quantum logic also chooses the second option when defining quantum 'negation', or *nequation*, as Fine calls it, where the 'q' reminds us of quantum theory and the difference in spelling helps us to keep in mind the difference between negation and nequation. The features of 'circular logic' hence have their corresponding analogues in quantum 'logic'. Let us consider them.

In quantum logic the elementary sentences are of the form 'observable A takes a value in the Borel set $\mathbb{b}$' – what we have been calling quantum events $P^A(\mathbb{b})$. For operators with discrete spectrum, the elementary sentences are of the form 'observable A takes a value a_i', where a_i is an eigenvalue of the operator A, and are represented by the projector $P^A_{a_i}$. (Note that the sentences or events are referred to a fixed system.) The assignments of truth values are simply the various states ψ of the system.[19] Indeed, for an elementary sentence P_{a_i} and state ψ

> P_{a_i} is true under an assignment V_ψ (i.e. in state ψ) iff
>
> ψ is an eigenstate of P_{a_i} [20]

The unary functor $\perp$ is defined on the quantum event $P \in \mathcal{L}(\mathcal{H})$ as

> $P^\perp \equiv$ orthogonal projection onto the complement of the closed subspace spanned by the range of P

An assignment V_ψ of truth values to the elementary sentences automatically assigns truth values to the nequation of P_{a_i}, i.e. $(P_{a_i})^\perp$, according to the rule

> If 'P_{a_i}' is true under V_ψ, then '$(P_{a_i})^\perp$' is false under V_ψ

Similarly to '$\neg$' in the 'circular logic', nequation '$\perp$' cannot be interpreted as logical negation nor an extension of it. For example, consider a two dimensional Hilbert space and an observable A with a discrete and non-degenerate spectrum $A = a_1 P_{a_1} + a_2 P_{a_2}$. For P_{a_1} and P_{a_2} (which are orthogonal projectors), nequation prescribes that to deny that event P_{a_1}, i.e. observable A takes value a_1, is the case is to assert that event P_{a_2}, i.e. observable A takes value a_2, is the case. But again this does not seem to make sense. For the event could be any of the possible combinations of P_{a_1} and P_{a_2}, i.e. $c_1 P_{a_1} + c_2 P_{a_2}$ with $b \neq 0$ and $|c_1|^2 + |c_2|^2 = 1$, in which case A would simply take no value for state ψ.

Moreover, whereas the above semantic rule holds for nequation '$\perp$', its converse – while true for ordinary negation – does not hold here. For example, suppose that the assignment V_ψ derives from A taking no value, e.g. $\psi =$

[19] For ease of exposition we will stick to pure states ψ and operators with discrete spectrum.

[20] We use V_ψ to note an assignment of truth values rather than L_ψ, which can be confused with subspace L_ψ.

$c_1\alpha_1 + c_2\alpha_2$, with c_1 and c_2 different from zero. If P_{a_j} is the sentence 'A takes value a_j', then both 'A takes value a_j' and 'A takes a value a_i different from a_j' are false under V_ψ (because ψ is not an eigenstate of either). That is, both 'P_{a_j}' and '$(P_{a_j})^\perp$' are false under V_ψ and thus the semantic rule 'if '$(P_{a_i})^\perp$' is true under V_ψ, then 'P_{a_i}' is false under V_ψ' does not hold. The trouble arises because nequation of (P_ψ), i.e. it is false that A takes no value, implies $(P_\psi)^\perp$, i.e. A takes no value, and not P_{a_i}, i.e. A takes a determinate value (any of the eigenvalues of A), as it would intuitively do if it could be interpreted as negation.

If one wanted to introduce ordinary negation, then one would define the negation of 'A takes the value a_i' as the assertion that 'either A takes no value or it takes a value corresponding to an eigenvalue different from a_i'. This negation is true under an assignment V_ψ just in case ψ is either not an eigenstate of A, i.e. ψ is a superposition of eigenstates of A with distinct eigenvalues, or ψ is an eigenstate of A but with eigenvalue different from a_i, i.e. ψ lies in the subspace $(L_{a_i})^\perp$ orthogonal to the space spanned by P_{a_i}. Thus the negation of 'A takes the value a_i' is true under V_ψ iff either ψ is a superposition of eigenstates of A with distinct eigenvalues, or V_ψ lies in $(L_{a_i})^\perp$.

Both alternatives of defining negation are perfectly meaningful and experimentally verifiable. Nevertheless, as we have seen, quantum logic does not use this last negation. It instead focuses on only one of the alternatives above and takes the quantum 'negation' to be nequation, and thus takes the 'negation' of 'A takes the value a_i' to be 'A takes a value corresponding to an eigenvalue different from a_i', which is true under V_ψ just in case ψ lies in $(L_{a_i})^\perp$. Notice that nequation corresponds to negation for compatible observables, but is completely different from it for incompatible events. Indeed, the nequation of 'A takes no value under V_ψ', i.e. $(P_\psi)^\perp$, is also 'A takes no value under V_ψ', and thus has nothing to do with negation. Hence, similarly to quantum conditional probability, the fact that nequation is co-extensive with negation in their shared domain of application, does not guarantee that outside that domain the nequation can be regarded as an extension or a generalization of negation.

Conjunction is defined in quantum 'logic' analogously to that of the 'circular logic':

> '$P_{a_i} \wedge P_{a_j}$' is true under V_ψ iff 'P_{a_i}' and 'P_{a_j}' are true under V_ψ.

As we can see, the functor '$\wedge$' is just the ordinary relation 'and'. Disjunction is also defined in quantum 'logic' analogously to disjunction in the 'circular logic':

> If 'P_{a_i}' is true under V_ψ or 'P_{a_j}' is true under V_ψ, then '$P_{a_i} \vee P_{a_j}$' is true under V_ψ

and thus presents analogue problems for its interpretation as an extension of the logical 'or'. Indeed, the converse of this semantic rule for disjunction does not hold; that is, the disjunction can be true although neither disjunct is true. For example, for an assignment of truth values V_ψ with $\psi = c_1\alpha_1 + c_2\alpha_2$ and observable $A = a_1 P_{a_1} + a_2 P_{a_2}$, the disjunct $P_{a_1} \vee P_{a_2}$ is true in ψ (because ψ is an eigenstate of P_I), while neither P_{a_1} nor P_{a_2} are true in ψ (because if both c_1 and c_2 are different from zero ψ is not an eigenstate of P_{a_1} nor P_{a_2}). And hence, for the system in state ψ, A does not take value a_1 nor does it take value a_2, yet A does take some value. This certainly precludes understanding disjunction '$\vee$' as an extension of the logical relation 'or'.

Another particularly relevant example is the following. For incompatible quantities A and B, the conjunction $P_{a_i} \wedge P_{b_j}$ is false under all truth assignments. For example, for a spin $\frac{1}{2}$ particle and an assignment of truth values V_ψ with $\psi = c_1\psi_{s_{+z}} + c_2\psi_{s_{-z}}$, the conjunct $P_{s_{+z}} \wedge P_{s_{+x}}$ is false for any c_1, c_2. And hence the spin $\frac{1}{2}$ particle can never take both a positive value of spin along the z-axis and a positive value of spin along the x-axis. This is the famous non-simultaneity of incompatible observables (in this example S_x and S_z). Similarly, the non-simultaneity of position and momentum of a quantum mechanical particle, i.e. the non-localizability of such a particle in arbitrary regions of both position and momentum, is a consequence of the fact that the conjunction $P_{\delta x} \wedge P_{\delta p}$ is false under all truth assignments.

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